DIFFERENTIAL EQUATIONS

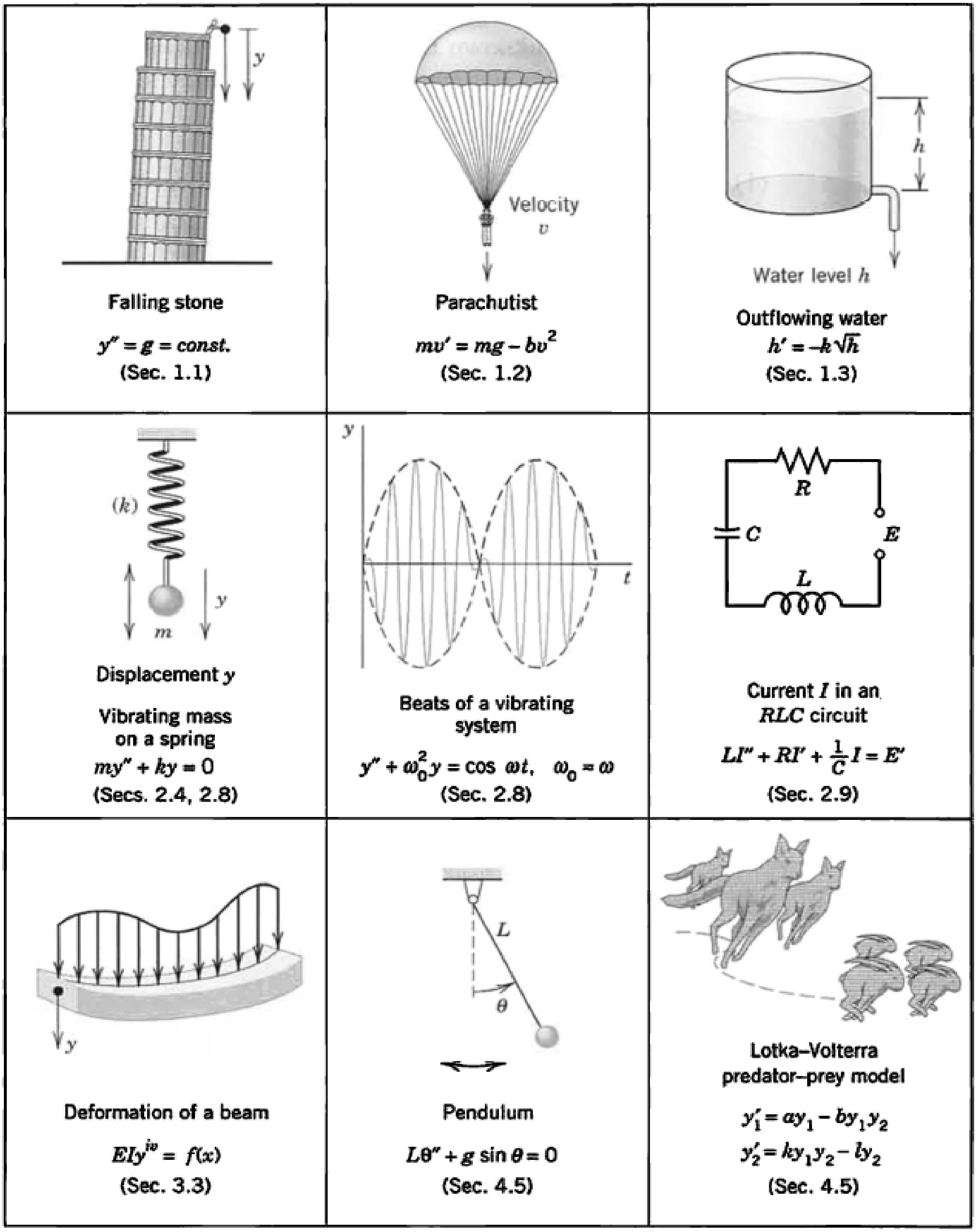

Fig. 1. Some applications of differential equations

DIFFERENTIAL EQUATIONS		NON-LINEAR	LINEAR (in y)	LINEAR W/ CST COEFFs (in y)
ORDINARY DIFF EQs	FIRST-ORDER	$4(y')^2 + x\cos y = x^2$	$4x^2y' + y\cos x = x^2$	$4y' + 3y = \cos x$
	SECOND-ORDER	$4y''y' + xy^{1/2} = x^2$	$4x^2y'' + yx^{1/2} = x^2$	$4y'' + 2y' + 3y = x^2$
	HIGHER-ORDER	$y^{(5)}y' \ldots \cos y''' \ldots y^{1/2}$	$y^{(5)} \ldots y''' \ldots y$	
PARTIAL DIFF EQs		$\frac{\partial y}{\partial x_1} \ldots \frac{\partial y}{\partial x_n}$		$4\frac{\partial y}{\partial x_1} + 3\frac{\partial y}{\partial x_2} = x_1 x_2$

FIRST-ORDER ODEs

Book: Kreyszig, 10th ed.

p4 (1.1)

general explicit:

$$y' = f(x,y) \Leftrightarrow \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \Leftrightarrow$$

$$P(x,y)dx + Q(x,y)dy = 0$$

p27 (1.5)

linear:

if $y' + p(x)y = q(x)$

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p12 (1.3)

separable:

if $P(x)dx + Q(y)dy = 0$

then integrate separately and resolve for y

general explicit: p4 (1.1)

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✓ separable: p12 (1.3)

if $P(x)dx + Q(y)dy = 0$

then integrate separately and resolve for y

✓ "linear" substitution:

if $\frac{dy}{dx} = f(ax + by + c)$

then use $\boldsymbol{u = ax + by + c}$

and solve separable eq. $\frac{du}{a + bf(u)} = dx$

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✓ simple "balanced" case:

p17 (mid)

if $y' = f(y/x)$,

that is $P(x,y)/Q(x,y) = -f(y/x)$

then use $\boldsymbol{y = xv}$

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generally "balanced" ("homogeneous" degree):

if $P(tx,ty) = t^\alpha P$ and $Q(tx,ty) = t^\beta Q$

and $\alpha = \beta$ (same degree of homogeneity)

then use $\boldsymbol{y = xv}$

and solve $\frac{1}{x^\alpha}\big((P + vQ)dx + xQdv\big) = 0$

i.e. $(P(1,v) + vQ(1,v))dx + xQ(1,v)dv = 0$

i.e. the separable eq. $\frac{dx}{x} = \left(\frac{P(1,v)}{Q(1,v)} + v\right)^{-1} dv$

FIRST-ORDER ODEs

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exact: (p20 (1.4))

if $\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$

then find $F(x,y)$ such that $\frac{\partial F}{\partial x} = \boldsymbol{P}$ and $\frac{\partial F}{\partial y} = \boldsymbol{Q}$,

integrate separately, and solution is $F(x,y) = C$

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exact with "integrating factor":

p23 (bott)

if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

then find $\boldsymbol{\mu(x)}$ such that $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$: possible **if** $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = g(x)$ only,

then $F(x,y)$ such that $\frac{\partial F}{\partial x} = \mu P$ and $\frac{\partial F}{\partial y} = \mu Q \Rightarrow F(x,y) = C$

p20 (1.4)

exact:

if $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$

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or conversely, $\mu(y)$ exists **if** $-\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = h(y)$ only exact:

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or conversely, $\boldsymbol{\mu(y)}$ exists **if** $-\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = h(y)$ only

or try $\boldsymbol{\mu(x,y) = x^m y^n}$ and solve $\dfrac{\partial \mu P}{\partial y} = \dfrac{\partial \mu Q}{\partial x}$ for m and n (no general condition)

p27 (1.5)

linear:

if $y' + p(x)y = q(x)$

p28 (top)

homogeneous linear:

if $\frac{dy}{dx} + p(x)y = 0$

("complementary equation")

then: $\ln y_c = -\int p(x)dx + C \Rightarrow y_c = \frac{C}{\alpha}$, = "***natural regime***"

where $\boldsymbol{\alpha} = \mathbf{exp}\left(\int \boldsymbol{p}(\boldsymbol{x})\boldsymbol{dx}\right)$ is the "integrating factor"

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the integrating factor $\boldsymbol{\alpha(x)}$ can also be found by setting:

$$\alpha\frac{dy}{dx} + \alpha p(x)y = \frac{d(\alpha y)}{dx} = \alpha q(x) \Rightarrow \frac{d\alpha}{dx} = \alpha(x)p(x)$$

with α, integrate $\frac{d(\alpha y)}{dx} = \alpha q \Rightarrow y = y_p + y_c$,

where $y_p = \frac{1}{\alpha}\left(\int \alpha(x)q(x)dx\right)$ is a "particular solution" = "*forced regime*"

p28 (bott)

FIRST-ORDER LINEAR ODEs

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p28 (bott)

equivalent method: "variation of the constant"

(find a particular solution by varying the coeff. of the homog. solution)

use $y_p = \frac{C(x)}{\alpha}$ and solve $\frac{dC}{dx} = \alpha q$

FIRST-ORDER LINEAR ODEs

linear:
if $y' + p(x)y = q(x)$
p27 (1.5)

homogeneous linear:
if $\frac{dy}{dx} + p(x)y = 0$
("complementary equation")
then: $\ln y_c = -\int p(x)dx + C \;\Rightarrow\; y_c = \frac{C}{\alpha}$, = "*natural regime*"
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p28 (top)

Bernoulli
if $\frac{dy}{dx} + p(x)y = q(x)\boldsymbol{y^N}$
then use $\boldsymbol{u(x)} = \boldsymbol{y^{1-N}}$ and solve linear eq.
$$\frac{du}{dx} + (1-N)p(x)u = (1-N)q(x)$$
p31 (bott)

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with α, integrate $\frac{d(\alpha y)}{dx} = \alpha q \;\Rightarrow\; y = y_p + y_c$,
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DIFFERENTIAL EQUATIONS		NON-LINEAR	LINEAR (in y)	LINEAR W/ CST COEFFs (in y)
ORDINARY DIFF EQs	FIRST-ORDER			
	SECOND-ORDER		Homogeneous Linear: $4x^2y'' + yx^{1/2} = 0$ NonHomogeneous Linear: $4x^2y'' + yx^{1/2} = x^2$	Homogeneous Linear CC: $4y'' + 2y' + 3y = 0$ NonHomogeneous Linear CC: $4y'' + 2y' + 3y = x^2$
	HIGHER-ORDER			
PARTIAL DIFF EQs				

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general (nonhomog.) linear

p79 (2.7)

if $y'' + P(x)y' + Q(x)y = R(x)$

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homogeneous linear:

p46 (2.1)

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("complementary equation")

superposition principle: if $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ are solutions,

then $\boldsymbol{y_c = C_1 y_1 + C_2 y_2}$ is a solution, too

- they form a basis for all solutions if they are linearly independent
- coefficients can be determined by initial conditions on *y* and *y'*

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"reduction of order":

p51 (2.1)

if one known solution is $y = y_1$

(found by "educated guess", or "ansatz")

then let the other be $\boldsymbol{y_2 = v(x) y_1}$ and solve

$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P$, i.e. $\ln v' = -2 \ln y_1 - \int P$

(also applies when P and Q are constant coefficients)

SECOND-ORDER LINEAR CONSTANT ODEs

general linear, constant coefficients:

if $ay'' + by' + cy = R(x)$

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p53 (2.2)

homogeneous linear, cst coefficients:

if $ay'' + by' + cy = 0$

then use $\boldsymbol{y = e^{rx}}$ and find roots r_1, r_2 of the "characteristic polynomial" $ar^2 + br + c = 0$:

- if $r_1 \neq r_2$, then $y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
- if $r_1 = r_2 = r$, then $y_c = C_1 e^{rx} + C_2 \boldsymbol{x} e^{rx}$

(if roots have imaginary part, rearrange expression into cos and sin)

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(if roots have imaginary part, rearrange expression into cos and sin)

with one "particular solution" $y = y_p$
(found by methods below)
then all solutions are given by: $\boldsymbol{y = y_p + y_c}$

finding a particular solution y_p

Rene Doursat

finding a particular solution y_p

“undetermined coefficients”: p81 (2.7)

(a) Basic Rule

- if $R(x) \sim \sin ax$ or $\cos ax$,
  try $y_p = A \sin ax + B \cos ax$
- if $R(x) \sim x^m$,
  try $y_p = A_m x^m + \cdots + A_1 x + A_0$
- if $R(x) \sim \exp(ax)$,
  try $y_p = C \exp(ax)$

(b) Modification Rule

- if y_p and y_c are linearly dependent,
  try a product $x^m y_p$

(c) Sum Rule

- if $R(x)$ is a sum or product of templates,
  try a sum or product of y_p templates

SECOND-ORDER LINEAR ODEs

general (nonhomog.) linear p79 (2.7)

if $y'' + P(x)y' + Q(x)y = R(x)$

homogeneous linear: p46 (2.1)

if $y'' + P(x)y' + Q(x)y = 0$

("complementary equation")

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(also applies when P and Q are constant coefficients)

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finding a particular solution y_p

finding a particular solution y_p

"variation of the constant":

p99 (2.10)

(find a particular solution by varying the coefficients of the homogeneous solution)

try $\boldsymbol{y_p = C_1(x) y_1 + C_2(x) y_2}$ such that

$$\begin{cases} y_1 C_1' + y_2 C_2' = 0 \\ y_1' C_1' + y_2' C_2' = R(x) \end{cases} \Rightarrow$$

then solve linear system for C_1' and C_2' (by Cramer's rule) and integrate

the determinant is called the "Wronskian":

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- $W \neq 0 \Leftrightarrow y_1, y_2$ linearly independent (most cases)
- $W'(x) = -P(x) W(x)$

p74 (2.6)

SECOND-ORDER LINEAR ODEs

general (nonhomog.) linear **p79 (2.7)**

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homogeneous linear: **p46 (2.1)**

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(also applies when P and Q are constant coefficients)

linear, missing y:

if $y'' + P(x)y' = R(x)$

then use $\boldsymbol{u = \frac{dy}{dx}}$ and solve first-order linear eq.

$u' + P(x)u = R(x)$ using α

with one "particular solution" $y = y_p$

(found by methods below)

then all solutions are given by: $\boldsymbol{y = y_p + y_c}$

finding a particular solution y_p

general explicit:
$y'' = F(y', y, x)$

↓

general, missing x:
if $y'' = F(y', y)$
then use $\boldsymbol{v} = \frac{dy}{dx}$ and try to solve for v
as a function of y, i.e. $\frac{dv}{dy} = \frac{1}{v} F(v, y)$, then integrate