

DIFFERENTIAL EQUATIONS

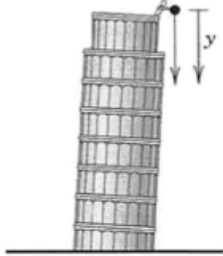
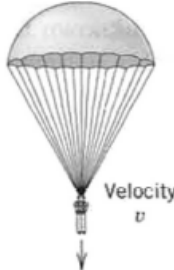
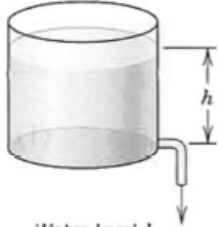
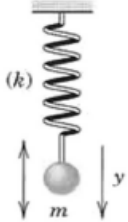
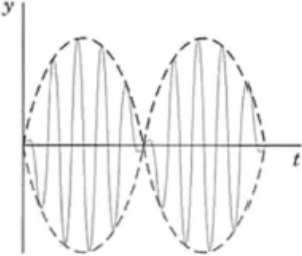
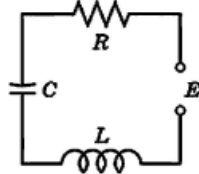
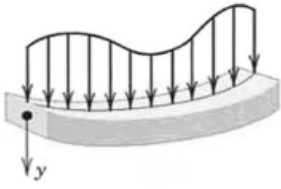
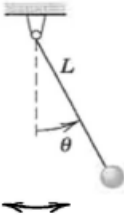
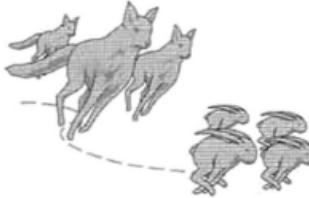
 <p>Falling stone $y'' = g = \text{const.}$ (Sec. 1.1)</p>	 <p>Parachutist $mv' = mg - bv^2$ (Sec. 1.2)</p>	 <p>Outflowing water $h' = -k\sqrt{h}$ (Sec. 1.3)</p>
 <p>Displacement y Vibrating mass on a spring $my'' + ky = 0$ (Secs. 2.4, 2.8)</p>	 <p>Beats of a vibrating system $y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 = \omega$ (Sec. 2.8)</p>	 <p>Current I in an RLC circuit $LI'' + RI' + \frac{1}{C}I = E'$ (Sec. 2.9)</p>
 <p>Deformation of a beam $Ely^{iv} = f(x)$ (Sec. 3.3)</p>	 <p>Pendulum $L\theta'' + g \sin \theta = 0$ (Sec. 4.5)</p>	 <p>Lotka-Volterra predator-prey model $y_1' = \alpha y_1 - \beta y_1 y_2$ $y_2' = k y_1 y_2 - l y_2$ (Sec. 4.5)</p>

Fig. 1. Some applications of differential equations

DIFFERENTIAL EQUATIONS		NON-LINEAR	LINEAR (in y)	LINEAR W/ CST COEFFs (in y)
ORDINARY DIFF EQs	FIRST- ORDER	$4(y')^2 + x \cos y = x^2$	$4x^2 y' + y \cos x = x^2$	$4y' + 3y = \cos x$
	SECOND- ORDER	$4y'' y' + x y^{1/2} = x^2$	$4x^2 y'' + y x^{1/2} = x^2$	$4y'' + 2y' + 3y = x^2$
	HIGHER- ORDER	$y^{(5)} y' \dots \cos y''' \dots y^{1/2}$	$y^{(5)} \dots y''' \dots y$	
PARTIAL DIFF EQs		$\frac{\partial y}{\partial x_1} \dots \frac{\partial y}{\partial x_n}$		$4 \frac{\partial y}{\partial x_1} + 3 \frac{\partial y}{\partial x_2} = x_1 x_2$

FIRST-ORDER ODEs

Book: Kreyszig, 10th ed.

general explicit:

p4 (1.1)

$$y' = f(x, y) \Leftrightarrow \frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} \Leftrightarrow$$
$$P(x, y)dx + Q(x, y)dy = 0$$

linear:

p27 (1.5)

$$\mathbf{if} \ y' + p(x)y = q(x)$$

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p12 (1.3)

$$\text{if } P(x)dx + Q(y)dy = 0$$

then integrate separately and resolve for y

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“linear” substitution:

$$\text{if } \frac{dy}{dx} = f(ax + by + c)$$

then use $u = ax + by + c$ and solve separable eq. $\frac{du}{a+bf(u)} = dx$

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simple “balanced” case:

p17 (mid)

if $y' = f(y/x)$,

that is $P(x, y)/Q(x, y) = -f(y/x)$

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generally “balanced” (“homogeneous” degree):

if $P(tx, ty) = t^\alpha P$ and $Q(tx, ty) = t^\beta Q$
 and $\alpha = \beta$ (same degree of homogeneity)
 then use $y = xv$

and solve $\frac{1}{x^\alpha} ((P + vQ)dx + xQdv) = 0$
 i.e. $(P(1, v) + vQ(1, v))dx + xQ(1, v)dv = 0$
 i.e. the separable eq. $\frac{dx}{x} = \left(\frac{P(1, v)}{Q(1, v)} + v\right)^{-1} dv$

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✓ exact: p20 (1.4)

if $\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}$

then find $F(x, y)$ such that $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$,
integrate separately, and solution is $F(x, y) = C$

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exact with “integrating factor”:
p23 (bott)

if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

then find $\mu(x)$ such that $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$: possible if $\frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = g(x)$ only,
 then $F(x, y)$ such that $\frac{\partial F}{\partial x} = \mu P$ and $\frac{\partial F}{\partial y} = \mu Q \Rightarrow F(x, y) = C$

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or conversely, $\mu(y)$ exists **if** $-\frac{1}{P} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = h(y)$ only

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or try $\mu(x, y) = x^m y^n$ and solve $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$
for m and n (no general condition)

linear:

p27 (1.5)

$$\text{if } y' + p(x)y = q(x)$$

homogeneous linear:

p28 (top)

$$\text{if } \frac{dy}{dx} + p(x)y = 0$$

("complementary equation")

$$\text{then: } \ln y_c = -\int p(x)dx + C \Rightarrow y_c = \frac{C}{\alpha}, = \text{"natural regime"}$$

where $\alpha = \exp(\int p(x)dx)$ is the "integrating factor"

linear: p27 (1.5)
if $y' + p(x)y = q(x)$

homogeneous linear: p28 (top)
if $\frac{dy}{dx} + p(x)y = 0$
 (“complementary equation”)
 then: $\ln y_c = -\int p(x)dx + C \Rightarrow y_c = \frac{C}{\alpha}$, = “*natural regime*”
 where $\alpha = \exp(\int p(x)dx)$ is the “integrating factor”

the integrating factor $\alpha(x)$ can also be found by setting:
 $\alpha \frac{dy}{dx} + \alpha p(x)y = \frac{d(\alpha y)}{dx} = \alpha q(x) \Rightarrow \frac{d\alpha}{dx} = \alpha(x)p(x)$

 with α , integrate $\frac{d(\alpha y)}{dx} = \alpha q \Rightarrow y = y_p + y_c$,
 where $y_p = \frac{1}{\alpha} (\int \alpha(x)q(x)dx)$ is a “particular solution” = “*forced regime*”
p28 (bott)

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p28 (bott)

equivalent method: “variation of the constant”
 (find a particular solution by varying the coeff. of the homog. solution)
 use $y_p = \frac{C(x)}{\alpha}$ and solve $\frac{dC}{dx} = \alpha q$

FIRST-ORDER LINEAR ODEs

linear: p27 (1.5)
if $y' + p(x)y = q(x)$

homogeneous linear: p28 (top)
if $\frac{dy}{dx} + p(x)y = 0$
 (“complementary equation”)
 then: $\ln y_c = -\int p(x)dx + C \Rightarrow y_c = \frac{C}{\alpha}$, = “*natural regime*”
 where $\alpha = \exp(\int p(x)dx)$ is the “integrating factor”

Bernoulli p31 (bott)
if $\frac{dy}{dx} + p(x)y = q(x)y^N$
 then use $u(x) = y^{1-N}$ and solve linear eq.
 $\frac{du}{dx} + (1-N)p(x)u = (1-N)q(x)$

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