Topology and Dynamics of Complex Networks


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Topology and Dynamics of Complex Networks

- Introduction
- Three structural metrics
- Four structural models
- Structural case studies
- Node dynamics and self-organization
- Bibliography
Topology and Dynamics of Complex Networks

• Introduction
  – Examples of complex networks
  – Elementary features
  – Motivations

• Three structural metrics

• Four structural models

• Structural case studies

• Node dynamics and self-organization

• Bibliography
# Introduction

Examples of complex networks – *Geometric, regular*

<table>
<thead>
<tr>
<th>Network</th>
<th>Nodes</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>BZ reaction</td>
<td>molecules</td>
<td>collisions</td>
</tr>
<tr>
<td>slime mold</td>
<td>amoebae</td>
<td>cAMP</td>
</tr>
<tr>
<td>animal coats</td>
<td>cells</td>
<td>morphogens</td>
</tr>
<tr>
<td>insect colonies</td>
<td>ants, termites</td>
<td>pheromone</td>
</tr>
<tr>
<td>flocking, traffic</td>
<td>animals, cars</td>
<td>perception</td>
</tr>
<tr>
<td>swarm sync</td>
<td>fireflies</td>
<td>photons ± long-range</td>
</tr>
</tbody>
</table>

- interactions inside a local neighborhood in 2-D or 3-D geometric space
- limited "visibility" within Euclidean distance
Introduction

Examples of complex networks – *Semi-geometric, irregular*

<table>
<thead>
<tr>
<th>Network</th>
<th>Nodes</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internet</td>
<td>routers</td>
<td>wires</td>
</tr>
<tr>
<td>brain</td>
<td>neurons</td>
<td>synapses</td>
</tr>
<tr>
<td>WWW</td>
<td>pages</td>
<td>hyperlinks</td>
</tr>
<tr>
<td>Hollywood</td>
<td>actors</td>
<td>movies</td>
</tr>
<tr>
<td>gene regulation</td>
<td>proteins</td>
<td>binding sites</td>
</tr>
<tr>
<td>ecology web</td>
<td>species</td>
<td>competition</td>
</tr>
</tbody>
</table>

- local neighborhoods (also) contain “long-range” links:
  - either “element” nodes located in space
  - or “categorical” nodes not located in space
- still limited “visibility”, but not according to distance
## Introduction

### Elementary features – *Node diversity & dynamics*

<table>
<thead>
<tr>
<th>Network</th>
<th>Node diversity</th>
<th>Node state/dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internet</td>
<td>routers, PCs, switches ...</td>
<td>routing state/algorithms</td>
</tr>
<tr>
<td>brain</td>
<td>sensory, inter, motor neuron</td>
<td>electrical potentials</td>
</tr>
<tr>
<td>WWW</td>
<td>commercial, educational ...</td>
<td>popularity, num. of visits</td>
</tr>
<tr>
<td>Hollywood</td>
<td>traits, talent</td>
<td>celebrity level, contracts</td>
</tr>
<tr>
<td>gene regulation</td>
<td>protein type, DNA sites ...</td>
<td>boundness, concentration</td>
</tr>
<tr>
<td>ecology web</td>
<td>species traits (diet, reprod.)</td>
<td>fitness, density</td>
</tr>
</tbody>
</table>

- Nodes can be of different subtypes: ⬤, ⬤, ⬤...
- Nodes have variable states of activity: ⬤ ⬤ ⬤ ⬤ ⬤
**Introduction**

**Elementary features – Edge diversity & dynamics**

<table>
<thead>
<tr>
<th>Network</th>
<th>Edge diversity</th>
<th>Edge state/dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internet</td>
<td>bandwidth (DSL, cable) ...</td>
<td>--</td>
</tr>
<tr>
<td>brain</td>
<td>excit., inhib. synapses ...</td>
<td>synap. weight, learning</td>
</tr>
<tr>
<td>WWW</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Hollywood</td>
<td>theater movie, TV series ...</td>
<td>partnerships</td>
</tr>
<tr>
<td>gene regulation</td>
<td>enhancing, blocking ...</td>
<td>mutations, evolution</td>
</tr>
<tr>
<td>ecology web</td>
<td>predation, cooperation</td>
<td>evolution, selection</td>
</tr>
</tbody>
</table>

- edges can be of different subtypes: 🔴, 🔵, 🔷 ...
- edges can also have variable weights: 🔐/∥∥∥∥∥
Introduction
Elementary features – *Network evolution*

- the **state** of a network generally evolves on two time-scales:
  - fast time scale: node activities
  - slow time scale: connection weights
- examples:
  - neural networks: activities & learning
  - gene networks: expression & mutations

- the **structural complexity** of a network can also evolve by adding or removing nodes and edges
- examples:
  - Internet, WWW, actors, ecology, etc.
Introduction
Motivations

✓ complex networks are the backbone of complex systems
  ▪ every complex system is a network of interaction among numerous smaller elements
  ▪ some networks are geometric or regular in 2-D or 3-D space
  ▪ other contain “long-range” connections or are not spatial at all
  ▪ understanding a complex system = break down into parts + reassemble

✓ network anatomy is important to characterize because structure affects function (and vice-versa)

✓ ex: structure of social networks
  ▪ prevent spread of diseases
  ▪ control spread of information (marketing, fads, rumors, etc.)

✓ ex: structure of power grid / Internet
  ▪ understand robustness and stability of power / data transmission
Topology and Dynamics of Complex Networks

• Introduction

• Three structural metrics
  – Average path length
  – Degree distribution (connectivity)
  – Clustering coefficient

• Four structural models

• Structural case studies

• Node dynamics and self-organization

• Bibliography
Three structural metrics

Average path length

- The path length between two nodes $A$ and $B$ is the smallest number of edges connecting them:

$$l(A, B) = \min l(A, A_i, \ldots, A_n, B)$$

- The average path length of a network over all pairs of $N$ nodes is

$$L = \langle l(A, B) \rangle = \frac{2}{N(N-1)} \sum_{A,B} l(A, B)$$

- The network diameter is the maximal path length between two nodes:

$$D = \max l(A, B)$$

- Property: $1 \leq L \leq D \leq N-1$
Three structural metrics
Degree distribution (connectivity)

- the degree of a node $A$ is the number of its connections (or neighbors), $k_A$
- the average degree of a network is
  \[ \langle k \rangle = \frac{1}{N} \sum_A k_A \]
- the degree distribution function $P(k)$ is the histogram (or probability) of the node degrees: it shows their spread around the average value

$0 \leq \langle k \rangle \leq N-1$

The degree of $A$ is 5
Three structural metrics
Clustering coefficient

- the neighborhood of a node $A$ is the set of $k_A$ nodes at distance 1 from $A$
- given the number of pairs of neighbors:
  $$F_A = \sum_{B, B'} 1 = k_A (k_A - 1) / 2$$
- and the number of pairs of neighbors that are also connected to each other:
  $$E_A = \sum_{B \leftrightarrow B'} 1$$
- the clustering coefficient of $A$ is
  $$C_A = E_A / F_A \leq 1$$
- and the network clustering coefficient:
  $$\langle C \rangle = 1/N \sum_A C_A \leq 1$$
Topology and Dynamics of Complex Networks

- Introduction
- Three structural metrics
  - Four structural models
    - Regular networks
    - Random networks
    - Small-world networks
    - Scale-free networks
- Structural case studies
- Node dynamics and self-organization
- Bibliography
Four structural models

Regular networks – *Fully connected*

- In a fully (globally) connected network, each node is connected to all other nodes.
- Fully connected networks have the LOWEST path length and diameter:
  \[ L = D = 1 \]
- The HIGHEST clustering coefficient:
  \[ C = 1 \]
- And a PEAK degree distribution (at the largest possible constant):
  \[ k_A = N-1, \quad P(k) = \delta(k - N+1) \]
- Also the highest number of edges:
  \[ E = N(N-1) / 2 \sim N^2 \]
Four structural models
Regular networks – *Lattice*

- A *lattice* network is generally structured against a geometric 2-D or 3-D background.
- For example, each node is connected to its nearest neighbors depending on the Euclidean distance:
  \[ A \leftrightarrow B \iff d(A, B) \leq r \]
- The radius \( r \) should be sufficiently small to remain far from a fully connected network, i.e., keep a large diameter:
  \[ D \gg 1 \]
Four structural models
Regular networks – *Lattice: ring world*

- In a *ring lattice*, nodes are laid out on a circle and connected to their $K$ nearest neighbors, with $K << N$.

- **HIGH average path length:**
  \[ L \approx \frac{N}{2K} \sim N \quad \text{for } N >> 1 \]
  (mean between closest node $l = 1$ and antipode node $l = N/K$)

- **HIGH clustering coefficient:**
  \[ C \approx 0.75 \quad \text{for } K >> 1 \]
  (mean between center with $K$ edges and farthest neighbors with $K/2$ edges)

- **PEAK degree distribution** (low value):
  \[ k_A = K, \quad P(k) = \delta(k - K) \]
Four structural models
Random networks

- **Random networks**
  - in a *random graph* each pair of nodes is connected with probability $p$
  - **LOW average path length**:
    \[ L \approx \frac{\ln N}{\ln \langle k \rangle} \sim \ln \langle k \rangle \quad \text{for } N \gg 1 \]
    (because the entire network can be covered in about $\langle k \rangle$ steps: $N \sim \langle k \rangle^L$)
  - **LOW clustering coefficient** (if sparse):
    \[ C = p = \frac{\langle k \rangle}{N} \ll 1 \quad \text{for } p \ll 1 \]
    (because the probability of 2 neighbors being connected is $p$, by definition)
  - **PEAK (Poisson) degree distribution** (low value):
    \[ \langle k \rangle \approx pN, \quad P(k) \approx \delta(k - pN) \]
Four structural models
Random networks

- Erdős & Rényi (1960): above a critical value of random connectivity the network is almost certainly connected in one single component.
- **Percolation** happens when “picking one button (node) will lift all the others”
- the critical value of probability $p$ is $p_c \approx \ln N / N$
- and the corresponding average critical degree: $\langle k_c \rangle \approx p_c N \approx \ln N$
Four structural models
Small-world networks

- A network with small-world EFFECT is ANY large network that has a low average path length:
  \[ L << N \quad \text{for} \quad N >> 1 \]

- famous “6 degrees of separation”

- the Watts-Strogatz (WS) small-world MODEL is a hybrid network between a regular lattice and a random graph

- WS networks have both the LOW average path length of random graphs:
  \[ L \sim \ln N \quad \text{for} \quad N >> 1 \]

- and the HIGH clustering coefficient of regular lattices:
  \[ C \approx 0.75 \quad \text{for} \quad K >> 1 \]
Four structural models
Small-world networks

Ring Lattice
- large world
- well clustered

Watts-Strogatz (1998)
- small world
- well clustered

Random graph
- small world
- poorly clustered

$p = 0$ (order)

$0 < p < 1$

$p = 1$ (disorder)

- the WS model consists in gradually rewiring a regular lattice into a random graph, with a probability $p$ that an original lattice edge will be reassigned at random.
Four structural models
Small-world networks

- the clustering coefficient is resistant to rewiring over a broad interval of \( p \)
  - it means that the small-world effect is hardly detectable locally: nodes continue seeing mostly the same "clique" of neighbors

- on the other hand, the average path length drops rapidly for low \( p \)
  - as soon as a few long-range "short-cut" connections are introduced, the original large-world starts collapsing
  - through a few bridges, far away cliques are put in contact and this is sufficient for a rapid spread of information
Four structural models
Small-world networks

- on the other hand, the WS model still has a PEAK (Poisson) degree distribution (uniform connectivity)
- in that sense, it belongs to the same family of exponential networks:
  - fully connected networks
  - lattices
  - random graphs
  - WS small-world networks
Four structural models
Scale-free networks

- in a scale-free network the degree distribution follows a POWER-LAW:
  \[ P(k) \sim k^{-\gamma} \]
- there exists a small number of highly connected nodes, called hubs (tail of the distribution)
- the great majority of nodes have few connections (head of the distribution)

A schematic scale-free network
Four structural models
Scale-free networks

- hyperbola-like, in linear-linear plot
- straight line, in log-log plot

\[ N = c \, A^{-b} \]

Typical aspect of a power law
(image from Larry Ruff, University of Michigan, http://www.geo.lsa.umich.edu/~ruff)
Four structural models
Scale-free networks

U.S. highway system
Random Network

U.S. airline system
Scale-Free Network

Bell Curve Distribution of Node Linkages
Power Law Distribution of Node Linkages

(Barabási & Bonabeau, 2003)
Four structural models
Scale-free networks

Regular networks are not resistant to random node failures: they quickly break down into isolated fragments.

Scale-free networks are remarkably resistant to random accidental node failures . . .

. . . however they are also highly vulnerable to targeted attacks on their hubs.

**Effect of failures and attacks on scale-free networks**
*(Barabási & Bonabeau, 2003)*
Four structural models

Scale-free networks

- In a random graph, the average path length increases significantly with node removal, then eventually breaks down.
  
  → For a while, the network becomes a large world.

- In a scale-free network, the average path length is preserved during random node removal.
  
  → It remains a small world.

- However, it fails even faster than a random graph under targeted removal.

(Albert, Jeong & Barabási, 1999)
Four structural models
Scale-free networks

- the **Barabási-Albert model**, reproduces the scale-free property by:
  - growth and
  - (linear) preferential attachment

- **growth**: a node is added at each step
- **attachment**: new nodes tend to prefer well-connected nodes ("the rich get richer" or "first come, best served") in linear proportion to their degree

*Growth and preferential attachment creating a scale-free network*
*(Barabási & Bonabeau, 2003)*
Topography and Dynamics of Complex Networks

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  – Internet
  – World Wide Web
  – Actors & scientists
  – Neural networks
  – Cellular metabolism

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Structural case studies

Internet

- the Internet is a network of routers that transmit data among computers
- routers are grouped into domains, which are interconnected
- to map the connections, “traceroute” utilities are used to send test data packets and trace their path

Schema of the Internet

(Wang, X. F., 2002)
Structural case studies
Internet

Map of Internet colored by IP address
(Bill Cheswick & Hal Burch, http://research.lumeta.com/ches/map)
Structural case studies
Internet

➢ the connectivity degree of a node follows a power of its rank (sorting out in decreasing order of degree):

\[ \text{node degree} \sim (\text{node rank})^{-\alpha} \]

➢ the most connected nodes are the least frequent:

\[ \text{degree frequency} \sim (\text{node degree})^{-\gamma} \]

\[ P(k) \sim k^{-\gamma} \]

→ the Internet is a scale-free network

Two power laws of the Internet topology
(Faloutsos, Faloutsos & Faloutsos, 1999)
Structural case studies

World Wide Web

- the World Wide Web is a network of documents that reference each other
- the nodes are the Web pages and the edges are the hyperlinks
- edges are directed: they can be outgoing and incoming hyperlinks

Schema of the World Wide Web of documents
Structural case studies
World Wide Web

Hierarchical topology of the international Web cache
Structural case studies
World Wide Web

WWW is a scale-free network:

\[ P(k) \sim k^{-\gamma} \]

with \( \gamma_{\text{out}} = 2.45 \) and \( \gamma_{\text{in}} = 2.1 \)

WWW is also a small world:

\[ L \approx \alpha \ln N \]

with \( L \approx 11 \) for \( N = 10^5 \) documents

Distribution of links on the World-Wide Web
(Albert, Jeong & Barabási, 1999)
Structural case studies
Actors & scientists

“The Oracle of Bacon”
http://www.cs.virginia.edu/oracle

- a given actor is on average 3 movies away from Kevin Bacon ($L_{\text{Bacon}} = 2.946$, as of June 2004) ...
  or any other actor for that matter

- Hollywood is a small world

- ... and it is a scale-free small world: a few actors played in a lot of movies, and a lot of actors in few movies

Path from K. Kline to K. Bacon = 3 (as of 1995)
(http://collegian.ksu.edu/issues/v100/FA/n069/fea-making-bacon-fuqua.html)
Structural case studies
Actors & scientists

“The Erdős Number Project”
http://www.oakland.edu/enp

Co-authors of Paul Erdős have number 1, co-authors of co-authors number 2, etc.

Mathematicians form a highly clustered small world ($C = 0.14$, $L = 7.64$)
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    - Node dynamics
    - Attractors in full & lattice networks
    - Synchronization in full networks
    - Waves in lattice networks
    - Epidemics in complex networks
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Node dynamics and self-organization

Node dynamics – *Individual node*

- each node in the network obey a differential equation: \( \frac{dx}{dt} = f(x) \)
- generally, three possible behaviors in phase space:
Node dynamics and self-organization

Node dynamics – *Coupled nodes*

- A complex network is a set of coupled nodes obeying:
  \[ \frac{dx_A}{dt} = f(x_A) + \sum_{A \leftarrow B} g(x_A, x_B) \]

- Generally, three types of complex network dynamics:
  - Fixed point node network
  - Limit cycle node network
  - Chaotic node network
Node dynamics and self-organization
Attractors in full networks

- fixed point nodes
- fully connected network
→ a few fixed patterns
(≈ 0.14 N)

Pattern retrieval in Hopfield memory:
full graph with Ising-type interactions
Node dynamics and self-organization
Attractors in lattice networks

Pattern formation in animal pigmentation:
2-D lattice with stationary reaction-diffusion
(NetLogo simulation, Uri Wilensky, Northwestern University, IL)
Node dynamics and self-organization

Synchronization in full networks

- limit cycle nodes
- fully connected network
  \( \rightarrow \text{global synchronization} \)

Spontaneous synchronization in a network of limit-cycle oscillators with distributed natural frequencies

\( \text{(Strogatz, 2001)} \)
Node dynamics and self-organization
Synchronization in full networks

Spontaneous synchronization in a swarm of fireflies:
(almost) fully connected graph of independent oscillators
(NetLogo simulation, Uri Wilensky, Northwestern University, IL)
Node dynamics and self-organization
Waves in lattice networks

BZ reaction or slime mold aggregation:
2-D lattice with oscillatory reaction-diffusion
(NetLogo simulation, Uri Wilensky, Northwestern University, IL)
Node dynamics and self-organization

Epidemics in complex networks

➢ understand of beneficial or nefarious activity/failures spread over a network:
  ▪ diseases
  ▪ power blackouts
  ▪ computer viruses
  ▪ fashions, etc.

➢ susceptible-infected-susceptible (SIS) epidemiological model:
  ▪ two node states: infected or susceptible
  ▪ susceptible nodes can get infected with probability $\nu$
  ▪ infected nodes heal and become susceptible again with proba $\delta$

$\rightarrow$ spreading rate: $\lambda = \nu / \delta$

3-D visualization of social links
Node dynamics and self-organization
Epidemics in complex networks

Exponential network → spread with threshold

Scale-free network → spread WITHOUT threshold

Epidemic on exponential and scale-free networks
(Pastor-Satorras & Vespignani, 2001)
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