Complex Networks 1
Small-World Models

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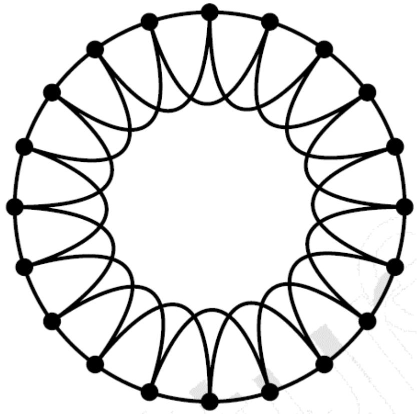
Outline

- Watts & Strogats 1998 – Collective dynamics of ‘small-world’ networks
- NetLogo: Small World
- Watts et al 2002 – Identity and search in social networks

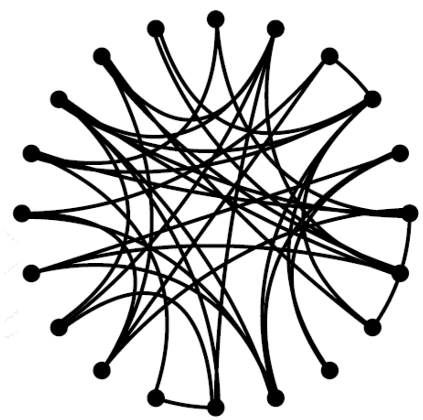
Typical Networks

- Typical examples of networks of coupled dynamical systems
  - Biological oscillators
  - Josephson junction arrays
  - Excitable media
  - Neural networks
  - Spatial games
  - Genetic control networks
- Ordinarily either regular or random connectivity

Regular



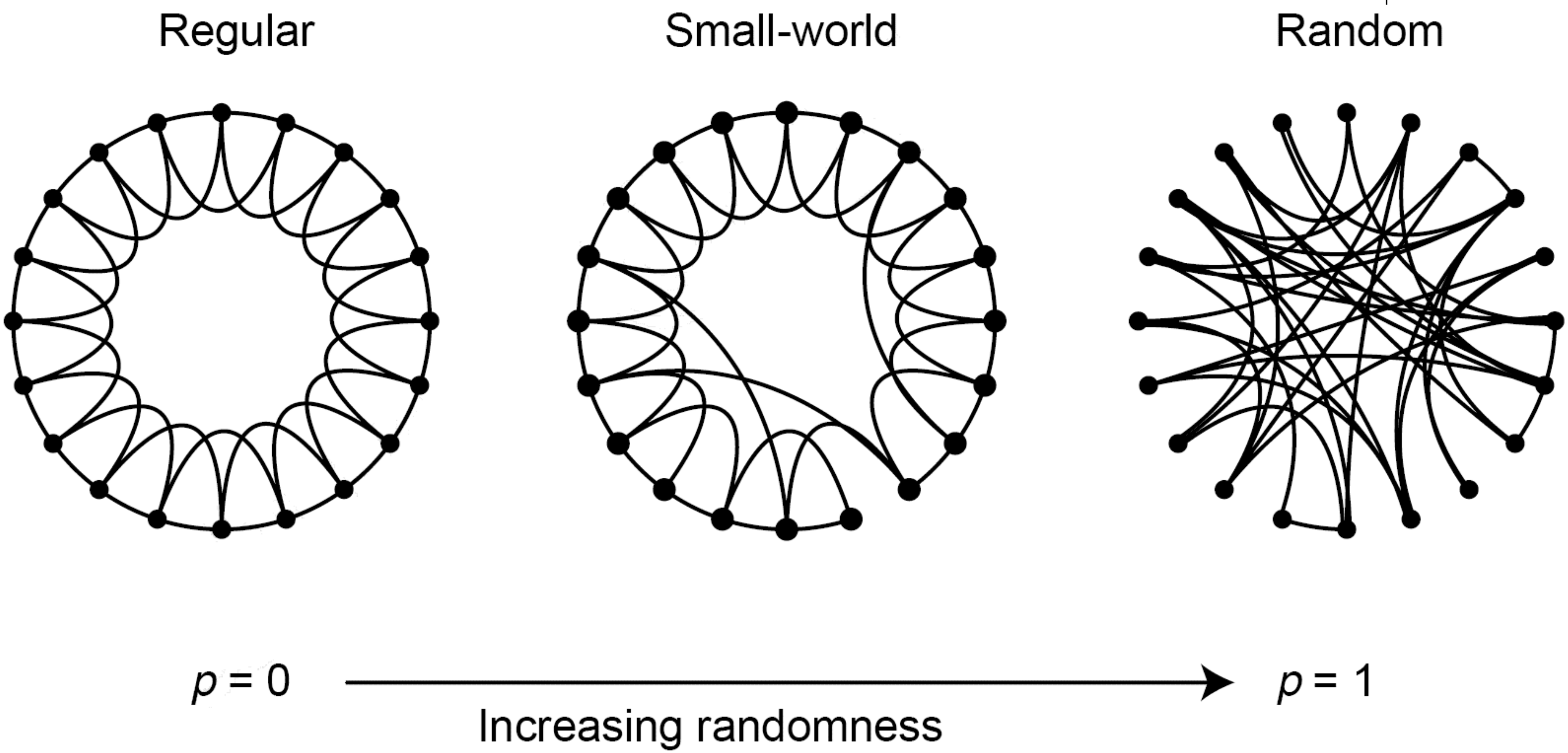
Random



Parameters

- Start with a ring lattice of n vertices and k edges per vertex
- Rewire each edge at random with probability p
  - $p = 0$ results in regularity
  - $p = 1$ results in complete randomness
- Quantify the structural properties by path length $L(p)$ and clustering coefficient $C(p)$
  - $L(p)$ is a global property, while $C(p)$ is local

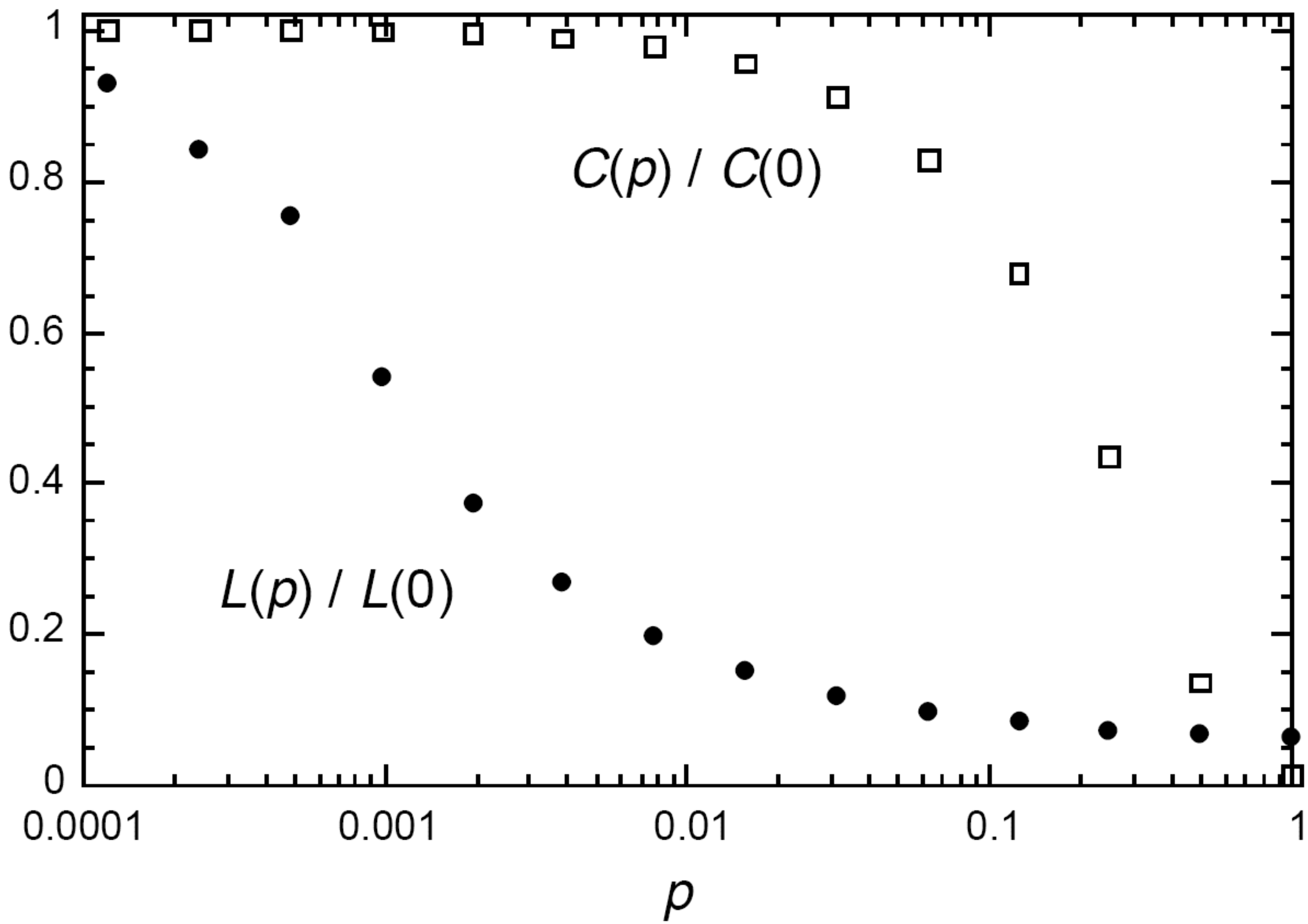
Ring Lattice Networks



Small World Networks - Behavior

- Desire many vertices with sparse connections, but enough to remain connected
- Require $n >> k >> \ln(n) >> 1$
  - $k >> \ln(n)$ guarantees that a random graph will be connected
  - $L \sim n/2k >> 1$ and $C \sim 3/4$ as $p \to 0$
  - $L \approx L_{\text{random}} \sim \ln(n)/\ln(k)$ and $C \sim C_{\text{random}} \sim k/n << 1$ as $p \to 1$

Small World Networks - Behavior



- n = 1,000, k = 10

Small World Networks - Examples

Table 1 Empirical examples of small-world networks

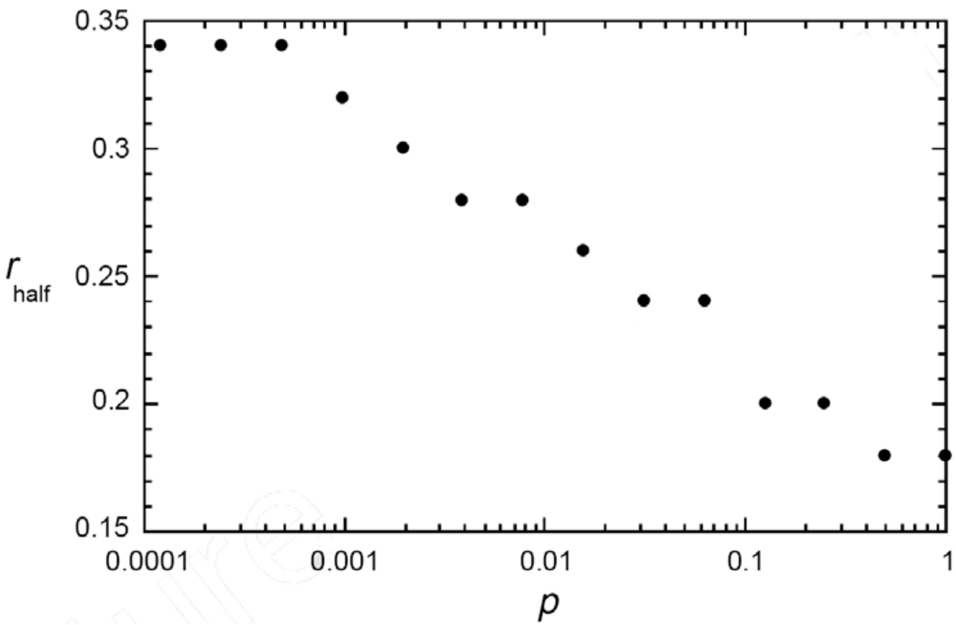
	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

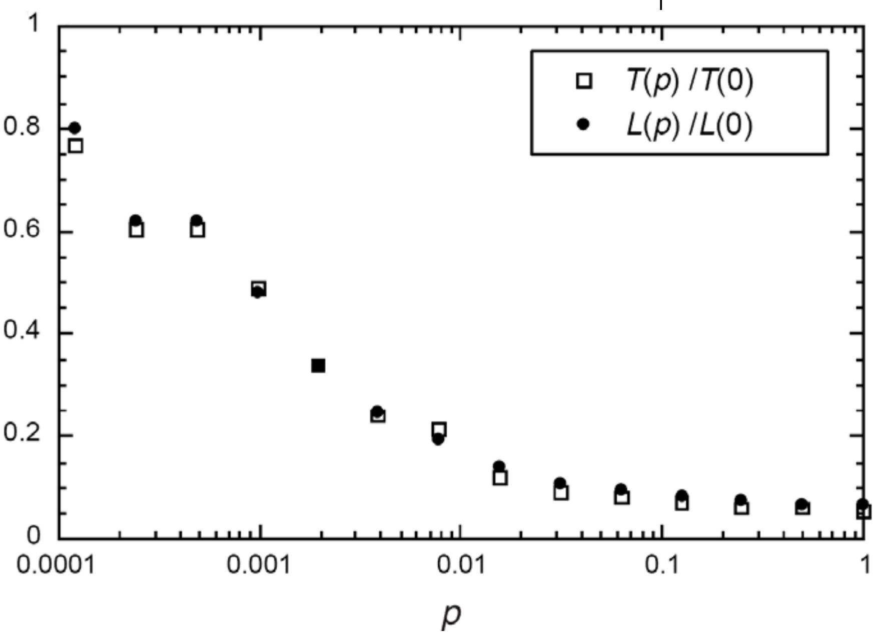
Characteristic path length L and clustering coefficient C for three real networks, compared to random graphs with the same number of vertices (n) and average number of edges per vertex (k). (Actors: $n = 225{,}226$, $k = 61$. Power grid: $n = 4{,}941$, $k = 2.67$. *C. elegans*: $n = 282$, $k = 14$.) The graphs are defined as follows. Two actors are joined by an edge if they have acted in a film together. We restrict attention to the giant connected component[16] of this graph, which includes ~90% of all actors listed in the Internet Movie Database (available at http://us.imdb.com), as of April 1997. For the power grid, vertices represent generators, transformers and substations, and edges represent high-voltage transmission lines between them. For *C. elegans*, an edge joins two neurons if they are connected by either a synapse or a gap junction. We treat all edges as undirected and unweighted, and all vertices as identical, recognizing that these are crude approximations. All three networks show the small-world phenomenon: $L \gtrsim L_{random}$ but $C \gg C_{random}$.

Small World Networks – Functional Significance

- Use a simplified model for the spread of an infectious disease using ring lattice networks
  - At time $t = 0$ a single infective individual is introduced
  - Infective individuals are removed permanently either by immunity or death after some unit time
  - Each infective individual can infect each of its neighbors with a probability r
  - Simulation runs until everyone is infected or the disease dies out

Small World Networks – Functional Significance





- The critical infectiousness r_{half} decreases rapidly as you increase p from a small value
- The time $T(p)$ required for global infection follows the $L(p)$ curve closely

Other Applications

- Density classification
  - Cellular automata used in this computational task with a ‘majority-rule’ running on a small-world graph can outperform all known human and genetic algorithm-generated rules running on a ring lattice
- Coupled phase oscillators
  - Small-world networks synchronize almost as readily as in the mean-field model, despite having orders of magnitude fewer edges

Other Applications

- Prisoner's dilemma
  - As the fraction of short cuts increases, cooperation is less likely
  - With increasing *p*, the likelihood of cooperative strategies evolving is reduced
- NetLogo

	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

PRISONER'S DILEMMA

The conundrum of the Prisoner's Dilemma is captured in the chart above. Defection (D) yields either 5 or 1, and cooperation yields either 3 or 0. The rational decision is to defect. If both prisoners are rational, since they both have the same information, both defect, yielding a payoff of 1 each. If both prisoners cooperate—if both remain silent—both will earn a 3, a considerably better outcome than that generated by the rational action.

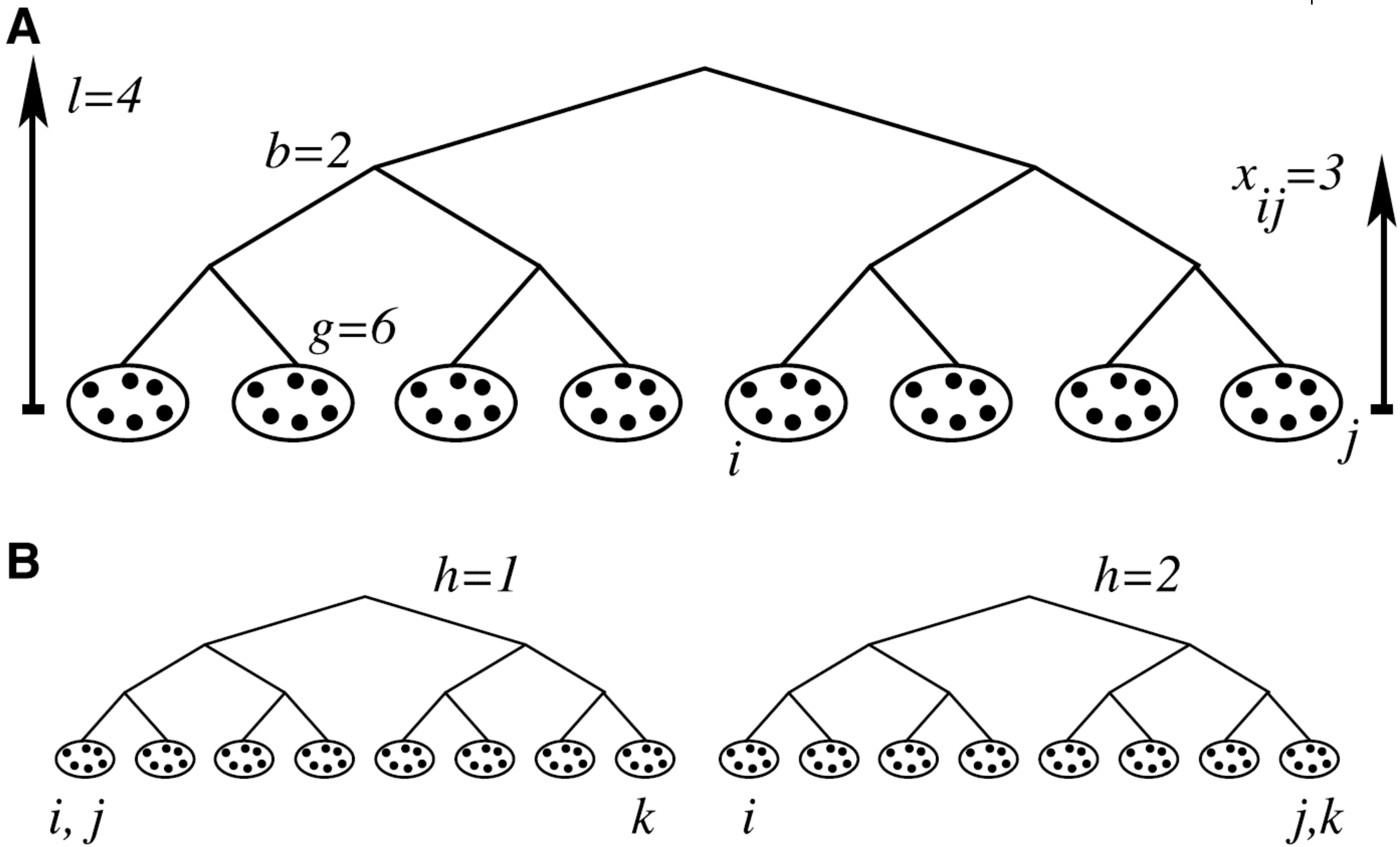
Searchable Networks

- Travers & Milgram (1969) demonstrated a real life social network with the property of being searchable
- This property has been seen in certain artificial networks
  - Networks that contain a certain fraction of highly connected nodes
  - Networks built on a geometric lattice acting as a proxy for social space
- Watts et al (2002) propose a socially plausible model that is also searchable

Six Contentions

1. Individuals have their own network ties and their own identities based on group associations
2. Individuals view the world as a hierarchy of layers each representing smaller and smaller groups as one goes down through the layers
    - An upper bound for group size for an individual’s own smallest group is around 100

Hierarchy (Construct, not Network)



Six Contentions

3. Group membership is a primary basis for social interaction
   - Probability of acquaintance between individuals *i* and *j* decreases with decreasing similarity of the groups they're in
   - $p(x) = c \exp[-\alpha x]$ where α is a tunable parameter and c is a normalizing constant
   - α is likened to homophily (like associates with like), so when $e^{-\alpha} << 1$, all links are as short as possible and when $e^{-\alpha} = b$ the network is uniformly random (similarity is irrelevant)

Six Contentions

4. Individuals hierarchically partition the social world in multiple ways (geography, occupation, etc.)
   - Categories are assumed to be independent
   - Each node's identity is defined as an *H*-dimensional coordinate vector $\boldsymbol{v}_i$, with each of the *H* components randomly assigned
   - The density of network ties must obey the constraint $z < g$ (number of friends is less than group size)

Six Contentions

5. Individuals construct a measure of social distance $y_{ij} = \min_h x^h_{ij}$
   - Based on perceived similarity with other nodes
   - Closeness in only a single dimension is sufficient to connote affiliation
6. As in Travers & Milgram (1969)'s example, each person forwards a message to a single neighbor based on limited information
   - Individual knows its own hierarchical vector $\boldsymbol{v}_i$, those of the immediate network neighbors $\boldsymbol{v}_j$, and that of the target $\boldsymbol{v}_t$

Efficient Search

- At each step in a chain from the original sender to the target there is a probability *p* that the message will terminate (~25% from Travers & Milgram (1969))
- The probability of an arbitrary message chain reaching the target is defined as
$q = \langle (1 - p)^L \rangle \geq r$
- Estimate maximum required average length
$\langle L \rangle \leq \ln r / \ln(1 - p)$

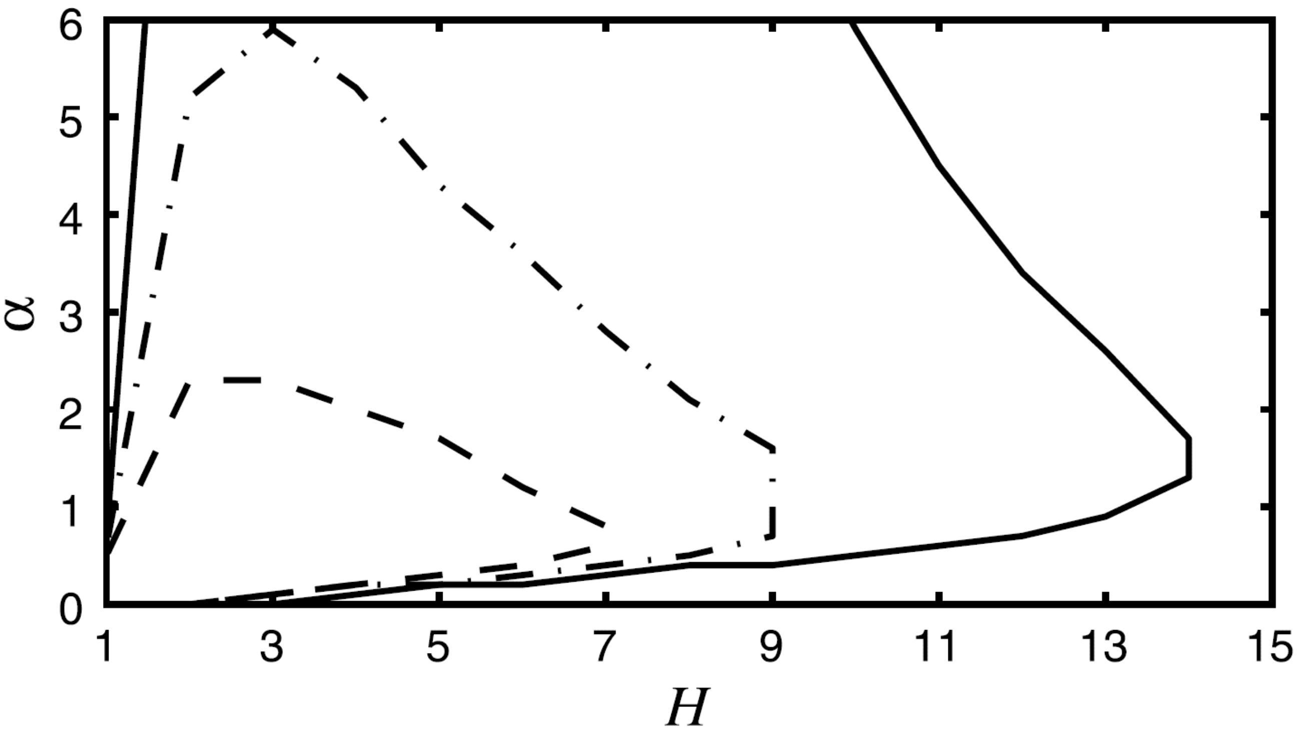
Searchable Networks

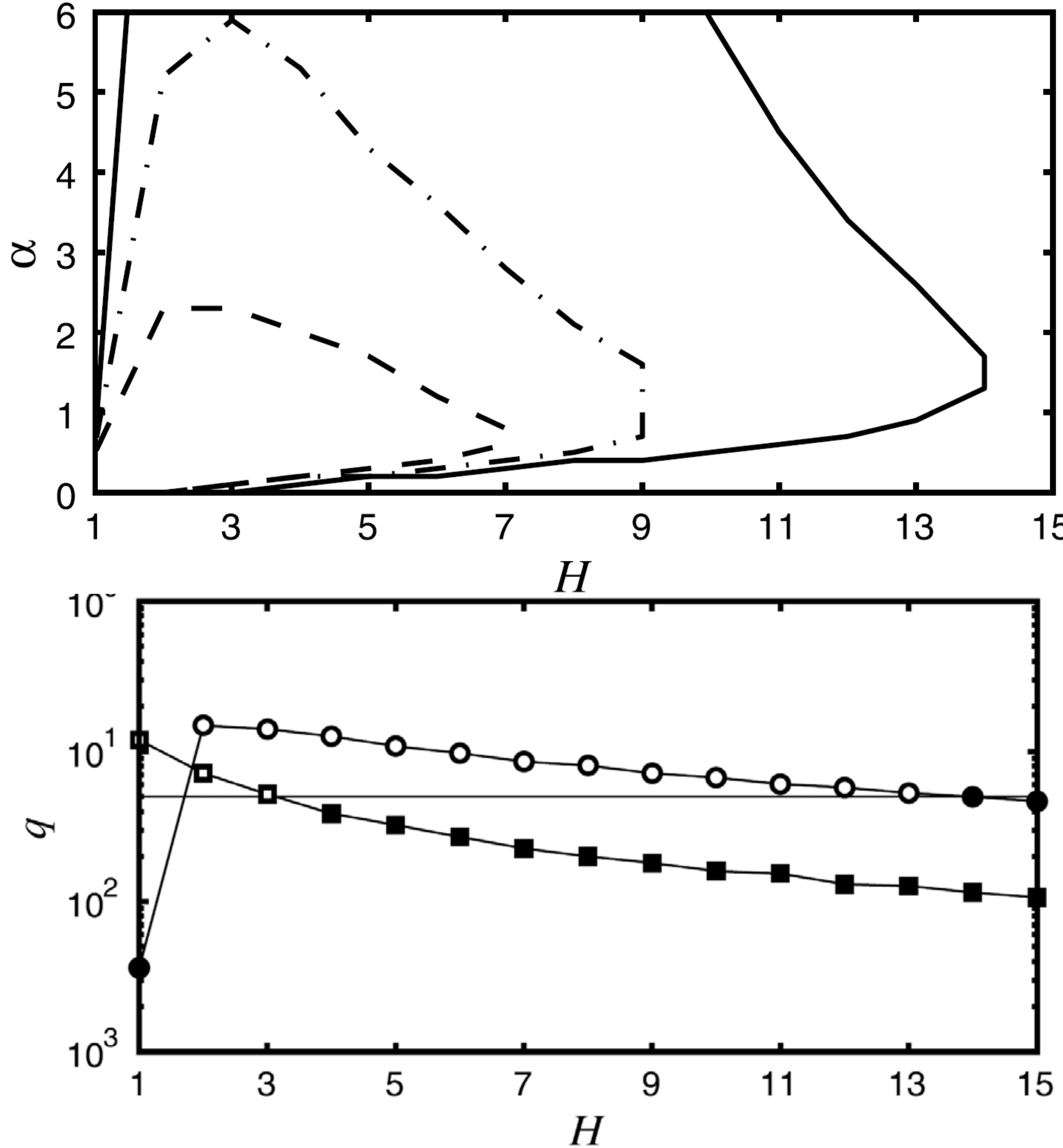
- For this study they set the values at $r = 0.05$ and $p = 0.25$, making $\langle L \rangle \leq 10.4$ independent of N
- $g = 100$
- $z = g - 1$
- N values:

102,400

204,800

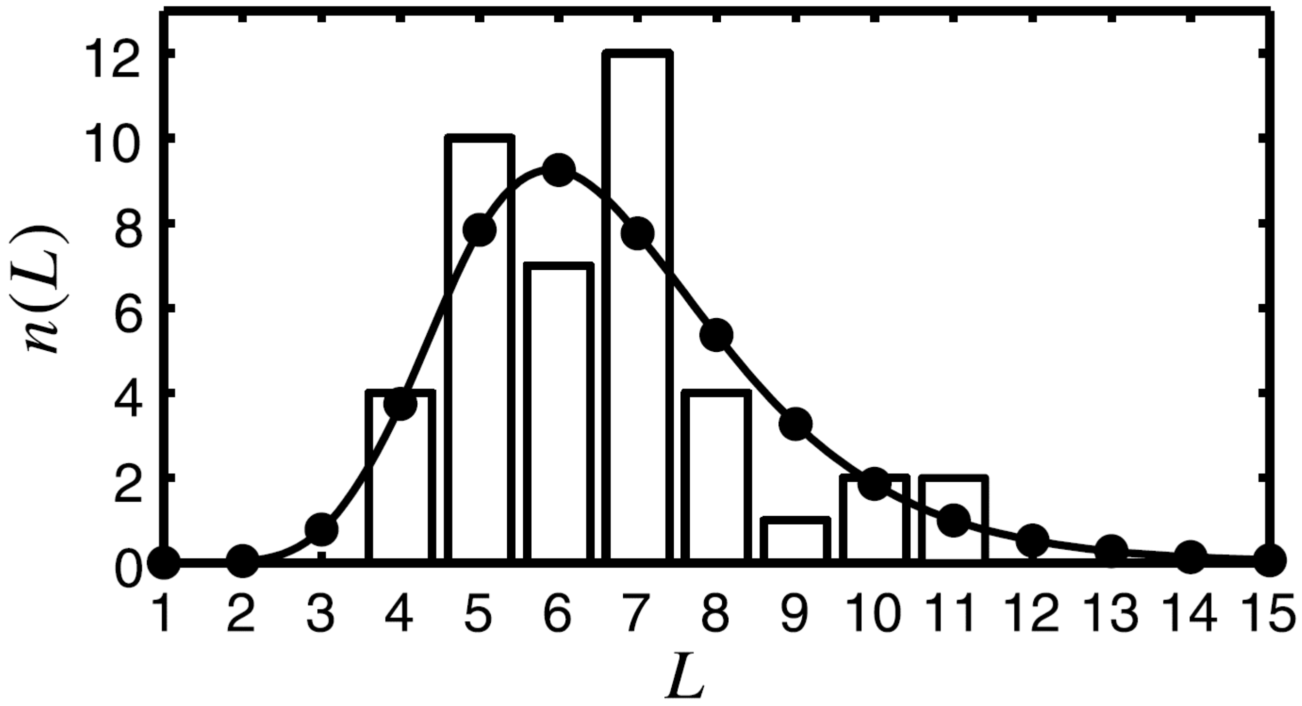
409,600





For N = 102,400 (solid line above)
α = 0 (squares)
α = 2 (circles)
Solid line at
r = 0.05

Comparison to Travers & Milgram (1969)



- Using the appropriate values $N = 10^8$, $p = 0.25$
- Values from empirical findings $z = 300$, $H = 2$
- Obtain $\langle L \rangle \approx 6.7$ for $\alpha = 1$ and $b = 10$