CS 790R Seminar
Modeling & Simulation

Topology and Dynamics of Complex Networks

~ Lecture 3: Review based on Strogatz (2001), Barabási & Bonabeau (2003), Wang, X. F. (2002) ~

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Topology and Dynamics of Complex Networks

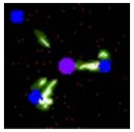
- Introduction
- Three structural metrics
- Four structural models
- Structural case studies
- Node dynamics and self-organization
- Bibliography

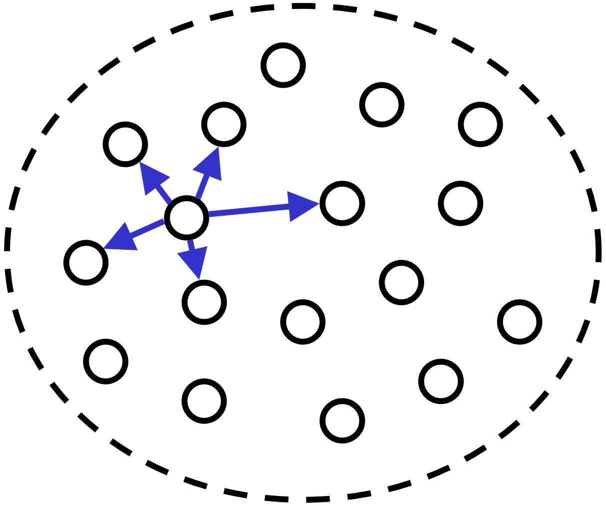
Topology and Dynamics of Complex Networks

- Introduction
  - Examples of complex networks
  - Elementary features
  - Motivations
- Three structural metrics
- Four structural models
- Structural case studies
- Node dynamics and self-organization
- Bibliography

Introduction

Examples of complex networks – *Geometric, regular*

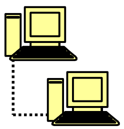
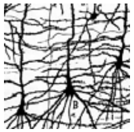
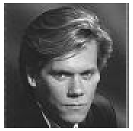
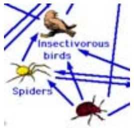
Network	*Nodes*	*Edges*
 BZ reaction	*molecules*	*collisions*
 slime mold	*amoebae*	*cAMP*
 animal coats	*cells*	*morphogens*
 insect colonies	*ants, termites*	*pheromone*
 flocking, traffic	*animals, cars*	*perception*
 swarm sync	*fireflies*	*photons* *± long-range*

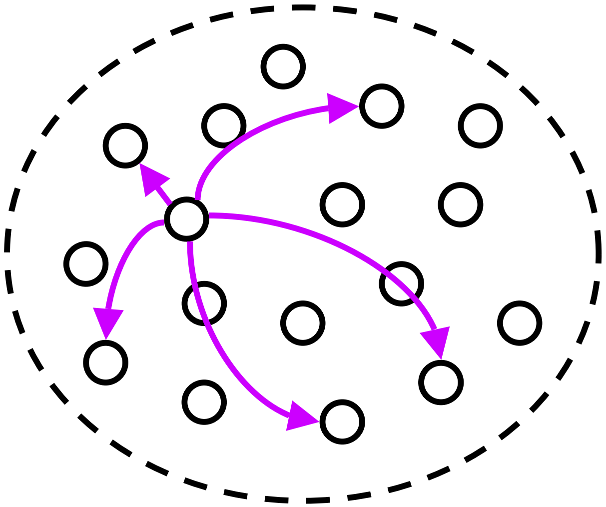


- interactions inside a local neighborhood in 2-D or 3-D geometric space
- limited "visibility" within Euclidean distance

Introduction

Examples of complex networks – *Semi-geometric, irregular*

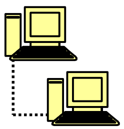
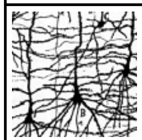
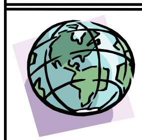
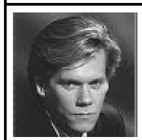
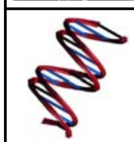
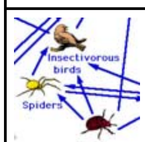
Network	*Nodes*	*Edges*
 Internet	*routers*	*wires*
 brain	*neurons*	*synapses*
 WWW	*pages*	*hyperlinks*
 Hollywood	*actors*	*movies*
 gene regulation	*proteins*	*binding sites*
 ecology web	*species*	*competition*

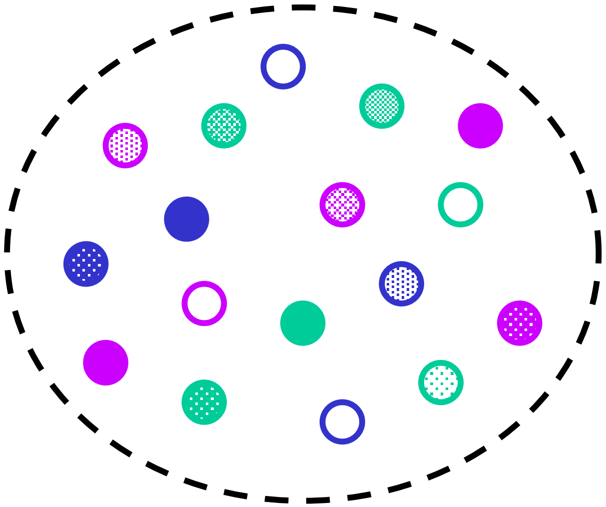


- local neighborhoods (also) contain "long-range" links:
  - either "element" nodes located in space
  - or "categorical" nodes not located in space
- still limited "visibility", but not according to distance

Introduction

Elementary features – *Node diversity & dynamics*

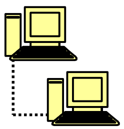
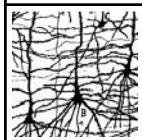
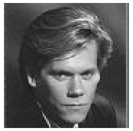
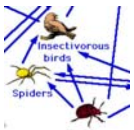
Network	*Node diversity*	*Node state/ dynamics*
 Internet	*routers, PCs, switches ...*	*routing state/ algorithm*
 brain	*sensory, inter, motor neuron*	*electrical potentials*
 WWW	*commercial, educational ...*	*popularity, num. of visits*
 Hollywood	*traits, talent ...*	*celebrity level, contracts*
 gene regulation	*protein type, DNA sites ...*	*boundness, concentration*
 ecology web	*species traits (diet, reprod.)*	*fitness, density*

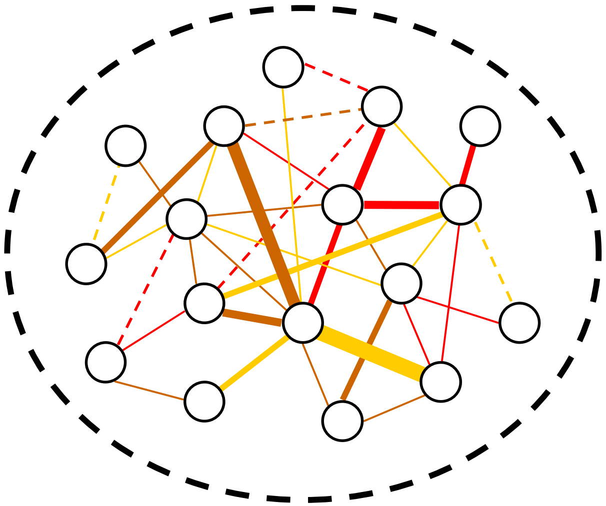


- nodes can be of different subtypes: ●, ●, ●...
- nodes have variable states of activity: ● ● ● ○

Introduction

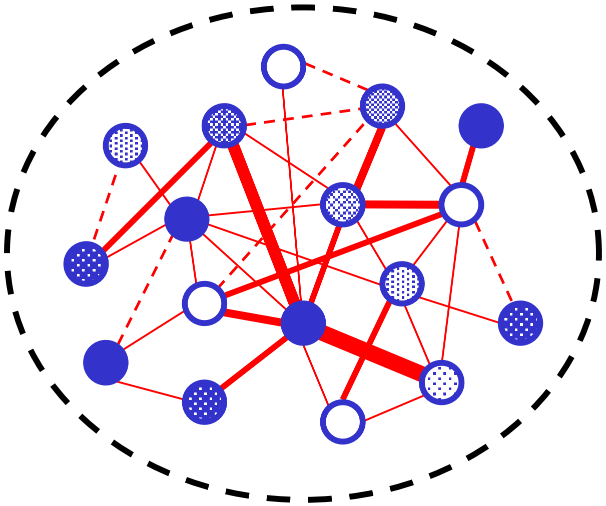
Elementary features – *Edge diversity & dynamics*

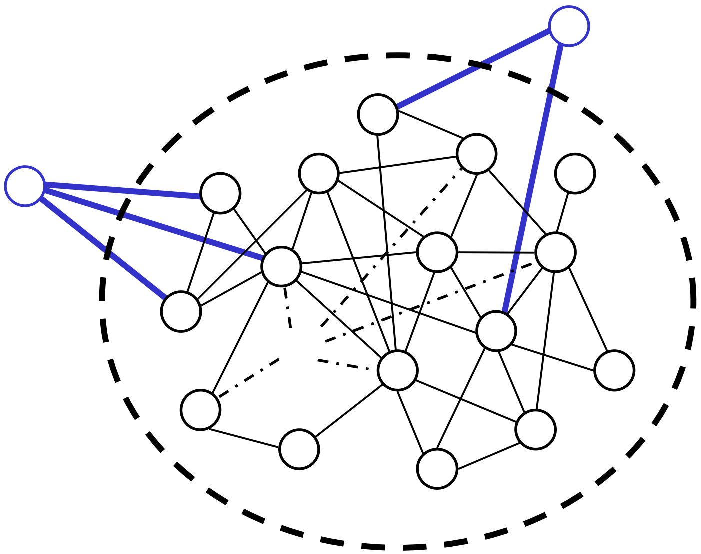
Network	*Edge diversity*	*Edge state/ dynamics*
 Internet	*bandwidth (DSL, cable)...*	*--*
 brain	*excit., inhib. synapses ...*	*synap. weight, learning*
 WWW	*--*	*--*
 Hollywood	*theater movie, TV series ...*	*partnerships*
 gene regulation	*enhancing, blocking ...*	*mutations, evolution*
 ecology web	*predation, cooperation*	*evolution, selection*



- edges can be of different subtypes: , , ...
- edges can also have variable weights:

Introduction
Elementary features – *Network evolution*





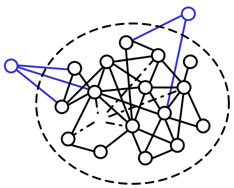
- the ***state*** of a network generally evolves on two time-scales:
  - fast time scale: node activities
  - slow time scale: connection weights
- examples:
  - neural networks: activities & learning
  - gene networks: expression & mutations

- the ***structural complexity*** of a network can also evolve by adding or removing nodes and edges
- examples:
  - Internet, WWW, actors. ecology, etc.

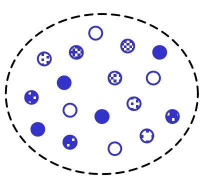
Introduction
Motivations

- ✓ complex networks are the backbone of complex systems
  - every complex system is a network of interaction among numerous smaller elements
  - some networks are geometric or regular in 2-D or 3-D space
  - other contain "long-range" connections or are not spatial at all
  - understanding a complex system = break down into parts + reassemble
- ✓ network anatomy is important to characterize because structure affects function (and vice-versa)
- ✓ ex: structure of social networks
  - prevent spread of diseases
  - control spread of information (marketing, fads, rumors, etc.)
- ✓ ex: structure of power grid / Internet
  - understand robustness and stability of power / data transmission

Topology and Dynamics of Complex Networks

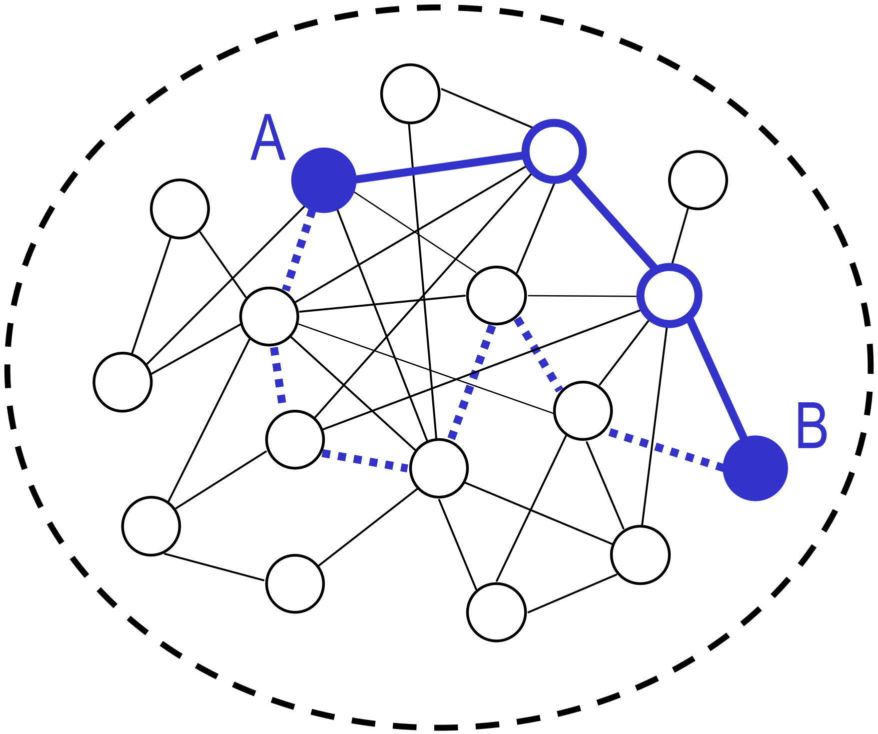


- Introduction
- Three structural metrics
  - Average path length
  - Degree distribution (connectivity)
  - Clustering coefficient
- Four structural models
- Structural case studies



- Node dynamics and self-organization
- Bibliography

Three structural metrics
Average path length



The path length between A and B is 3

- the *path length* between two nodes A and B is the smallest number of edges connecting them:

$$l(A, B) = \min l(A, A_i, \ldots A_n, B)$$

- the *average path length* of a network over all pairs of N nodes is

$$L = \langle l(A, B) \rangle$$

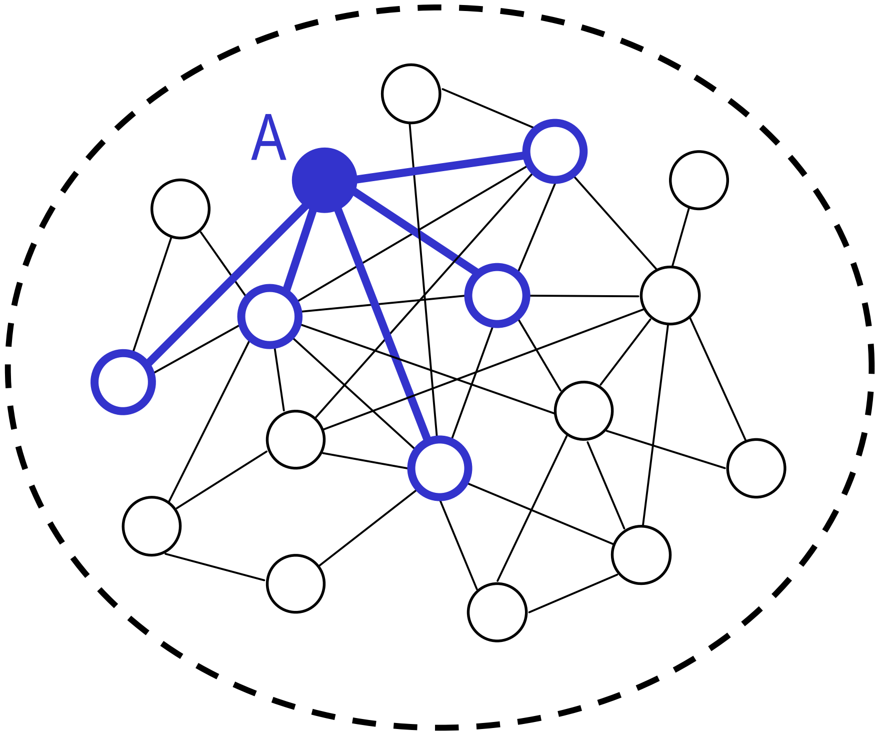
$$= 2/N(N-1) \sum_{A,B} l(A, B)$$

- the *network diameter* is the maximal path length between two nodes:

$$D = \max l(A, B)$$

- property: $1 \leq L \leq D \leq N-1$

Three structural metrics
Degree distribution (connectivity)



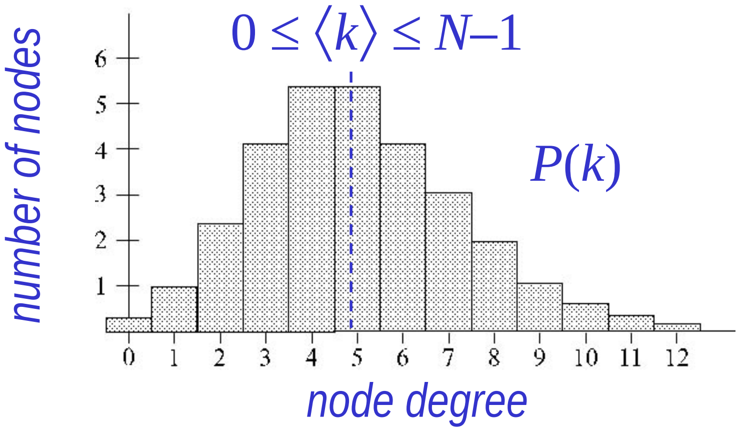
The degree of A is 5

- the *degree* of a node A is the number of its connections (or neighbors), k_A
- the *average degree* of a network is

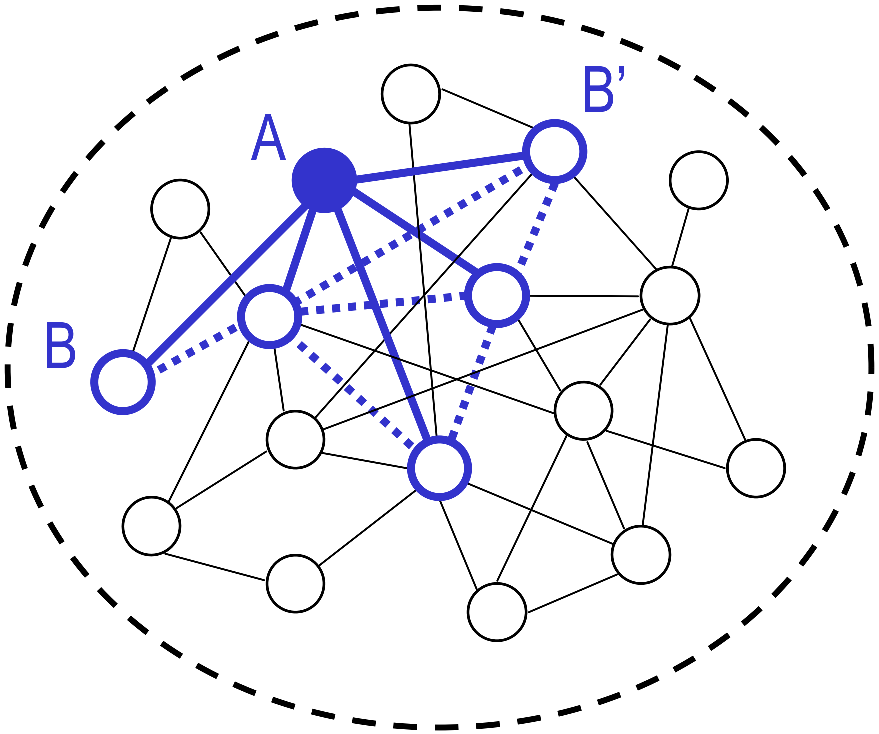
$$\langle k \rangle = 1/N \sum_A k_A$$

- the *degree distribution* function $P(k)$ is the histogram (or probability) of the node degrees: it shows their spread around the average value

$$0 \le \langle k \rangle \le N-1$$



Three structural metrics
Clustering coefficient



The clustering coefficient of A is 0.6

- the *neighborhood* of a node A is the set of k_A nodes at distance 1 from A
- given the number of *pairs* of neighbors:

$$F_A = \sum_{B,B'} 1 = k_A (k_A - 1) / 2$$

- and the number of pairs of neighbors that are also *connected* to each other:

$$E_A = \sum_{B \leftrightarrow B'} 1$$

- the *clustering coefficient* of A is

$$C_A = E_A / F_A \quad \leq 1$$

- and the *network clustering coefficient*:

$$\langle C \rangle = 1/N \sum_A C_A \quad \leq 1$$

Topology and Dynamics of Complex Networks

- Introduction
- Three structural metrics
- Four structural models
  - Regular networks
  - Random networks
  - Small-world networks
  - Scale-free networks
- Structural case studies
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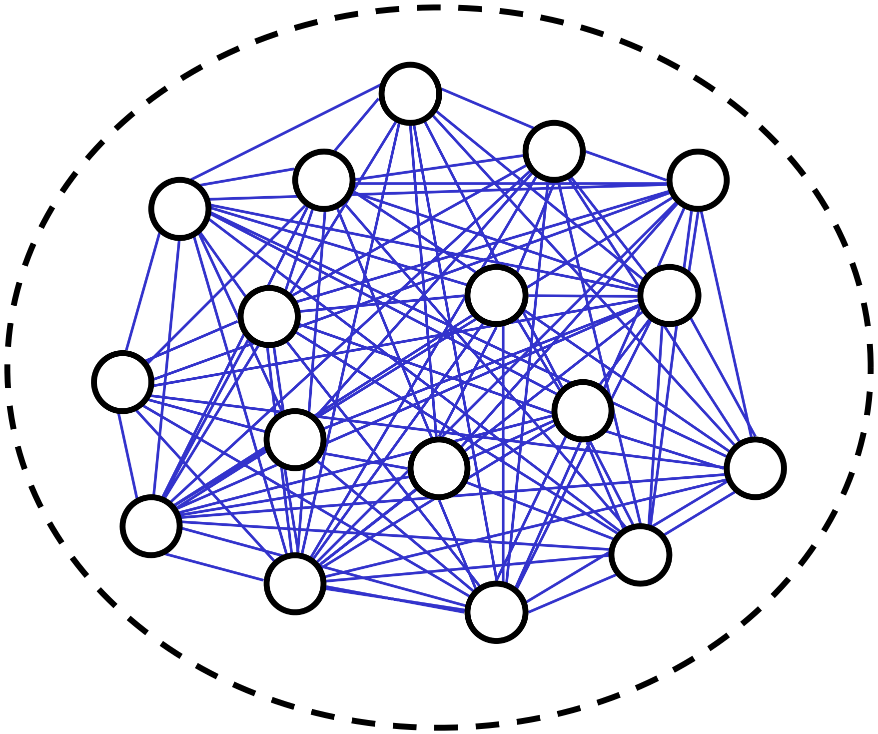
Four structural models

Regular networks – *Fully connected*



A fully connected network

- in a *fully (globally) connected* network, each node is connected to all other nodes
- fully connected networks have the *LOWEST path length* and diameter:

$$L = D = 1$$

- the *HIGHEST clustering coefficient*:

$$C = 1$$

- and a *PEAK degree distribution* (at the largest possible constant):

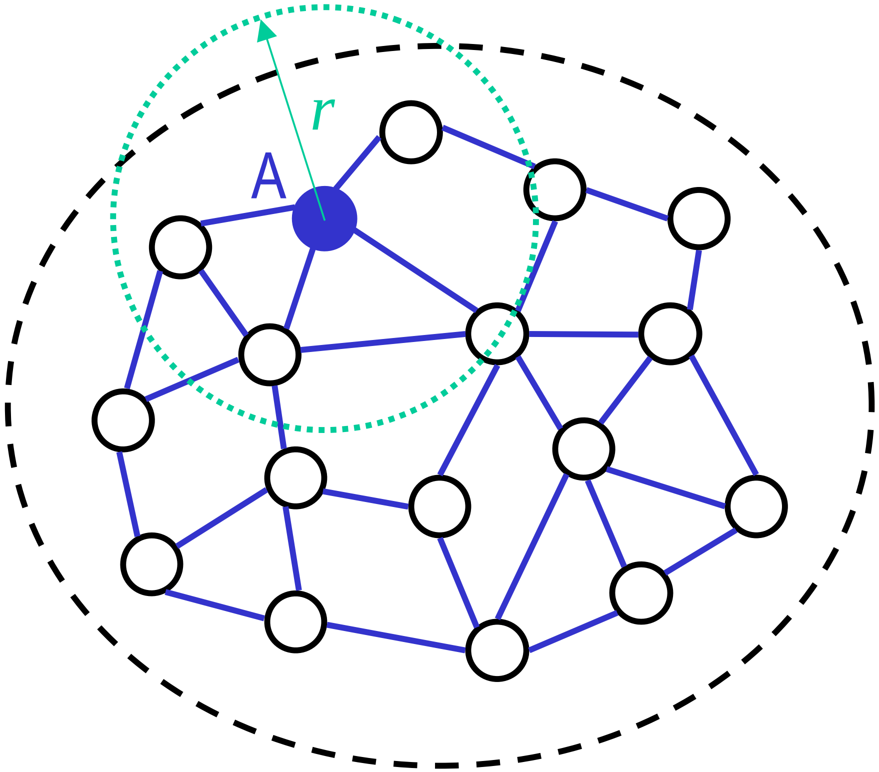
$$k_A = N{-}1, \qquad P(k) = \delta(k - N + 1)$$

- also the highest number of edges:

$$E = N(N{-}1)\,/\,2 \;\sim\; N^2$$

Four structural models

Regular networks – *Lattice*



A 2-D lattice network

- a *lattice* network is generally structured against a geometric 2-D or 3-D background
- for example, each node is connected to its nearest neighbors depending on the Euclidean distance:

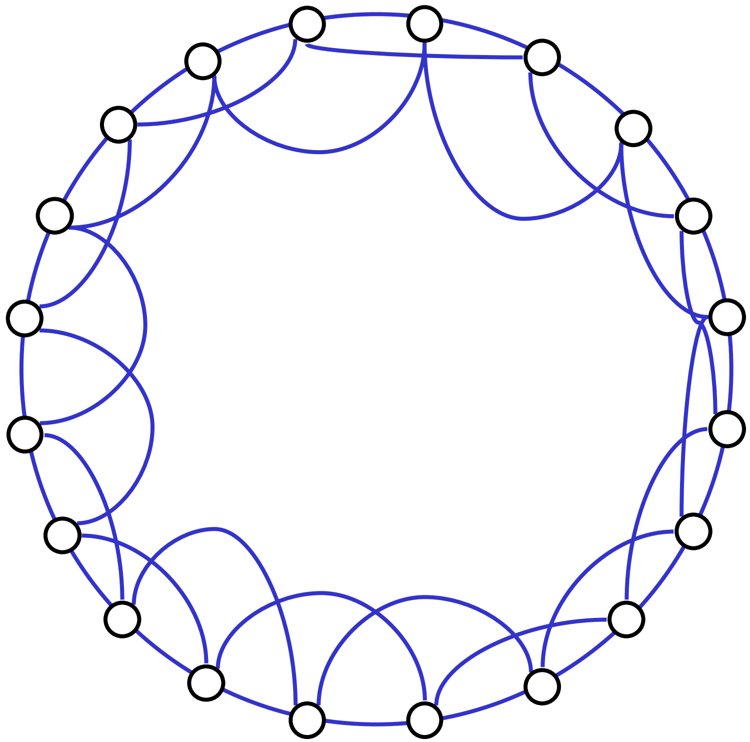
$$A \leftrightarrow B \iff d(A, B) \leq r$$

- the radius r should be sufficiently small to remain far from a fully connected network, i.e., keep a large diameter:

$$D >> 1$$

Four structural models

Regular networks – *Lattice: ring world*



A ring lattice with K = 4

- in a *ring lattice*, nodes are laid out on a circle and connected to their K nearest neighbors, with $K << N$
- *HIGH average path length*:

$$L \approx N/2K \quad \sim N \quad \text{for } N >> 1$$

(mean between closest node $l = 1$ and antipode node $l = N/K$)

- *HIGH clustering coefficient*:

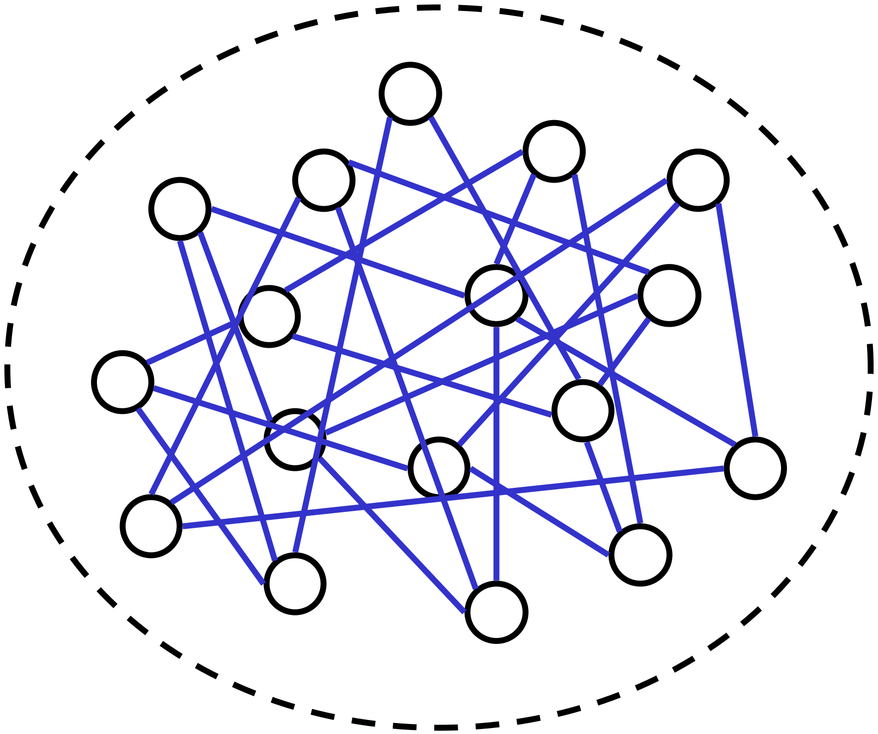
$$C \approx 0.75 \quad \text{for } K >> 1$$

(mean between center with K edges and farthest neighbors with $K/2$ edges)

- *PEAK degree distribution* (low value):

$$k_A = K, \quad P(k) = \delta(k - K)$$

Four structural models
Random networks



A random graph with $p = 3/N = 0.18$

- in a *random graph* each pair of nodes is connected with probability p
- *LOW average path length*:

  $L \approx \ln N / \ln\langle k \rangle \sim \ln N$ for $N >> 1$

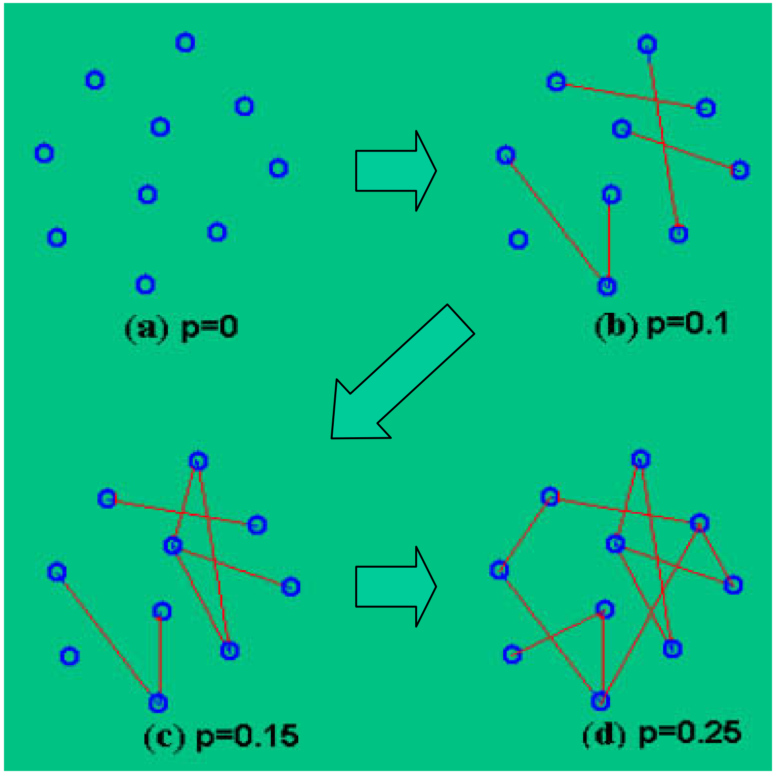
  (because the entire network can be covered in about $\langle k \rangle$ steps: $N \sim \langle k \rangle^L$)
- *LOW clustering coefficient* (if sparse):

  $C = p = \langle k \rangle / N << 1$ for $p << 1$

  (because the probability of 2 neighbors being connected is p, by definition)
- *PEAK (Poisson) degree distribution* (low value):

  $\langle k \rangle \approx pN, \quad P(k) \approx \delta(k - pN)$

Four structural models
Random networks



Percolation in a random graph

(Wang, X. F., 2002)

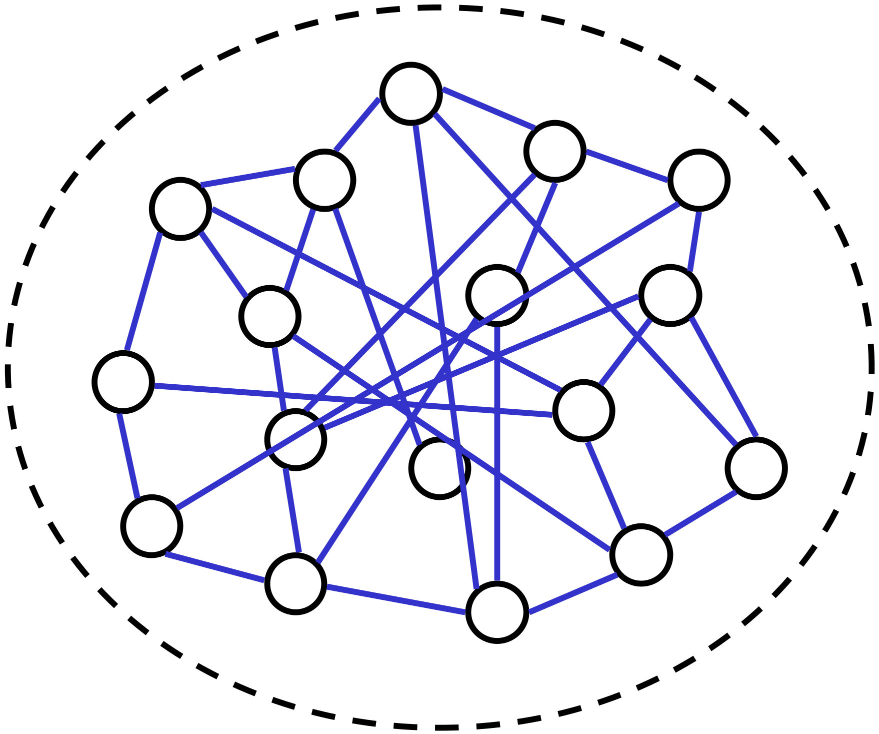
- Erdős & Rényi (1960): above a critical value of random connectivity the network is almost certainly connected in one single component
- *percolation* happens when "picking one button (node) will lift all the others"
- the critical value of probability p is

$$p_c \approx \ln N / N$$

- and the corresponding average critical degree:

$$\langle k_c \rangle \approx p_c N \approx \ln N$$

Four structural models
Small-world networks



A Watts-Strogatz small-world network

- a network with *small-world EFFECT* is ANY large network that has a low average path length:

$$L << N \quad \text{for } N >> 1$$

- famous "6 degrees of separation"
- the *Watts-Strogatz (WS) small-world MODEL* is a hybrid network between a regular lattice and a random graph
- WS networks have both the LOW average path length of random graphs:

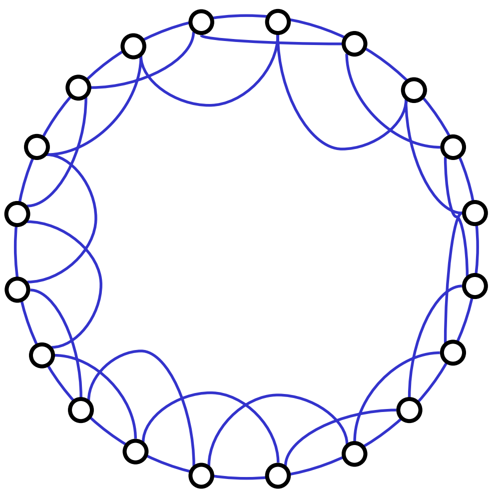
$$L \sim \ln N \quad \text{for } N >> 1$$

- and the HIGH clustering coefficient of regular lattices:

$$C \approx 0.75 \quad \text{for } K >> 1$$

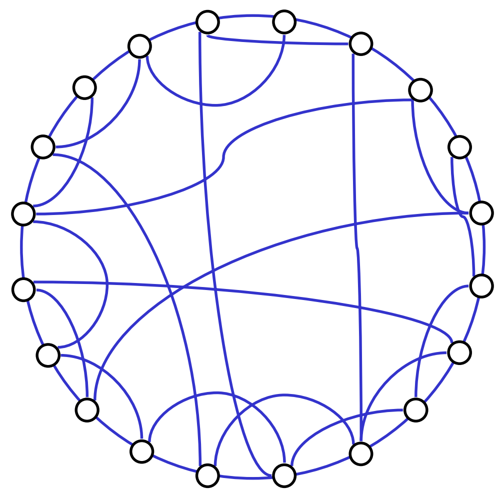
Four structural models
Small-world networks

Ring Lattice
- large world
- well clustered



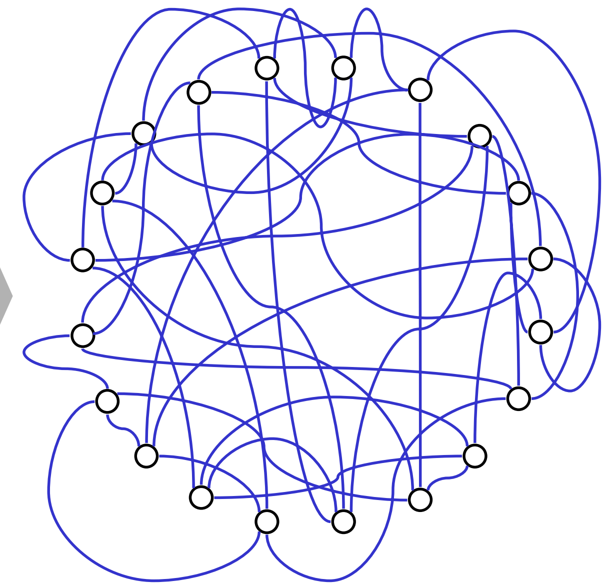
$p = 0$ (order)

Watts-Strogatz (1998)
- small world
- well clustered



$0 < p < 1$

Random graph
- small world
- poorly clustered

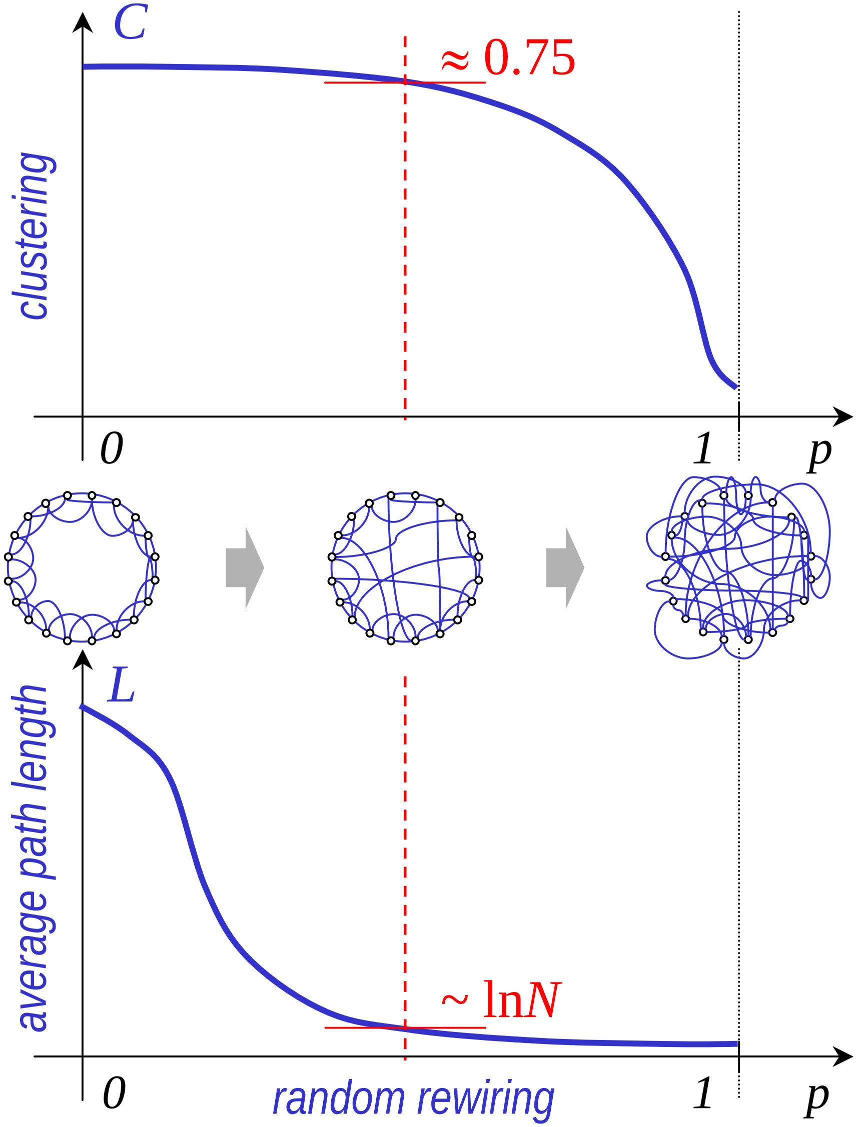


$p = 1$ (disorder)

➢ the WS model consists in gradually rewiring a regular lattice into a random graph, with a probability p that an original lattice edge will be reassigned at random

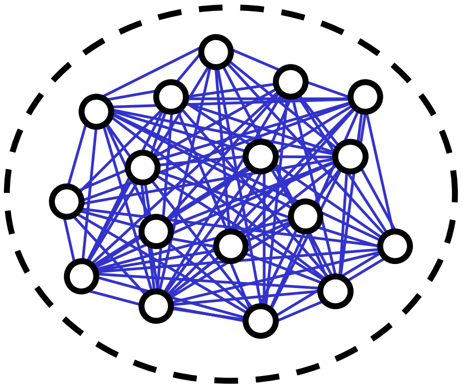
Four structural models

Small-world networks

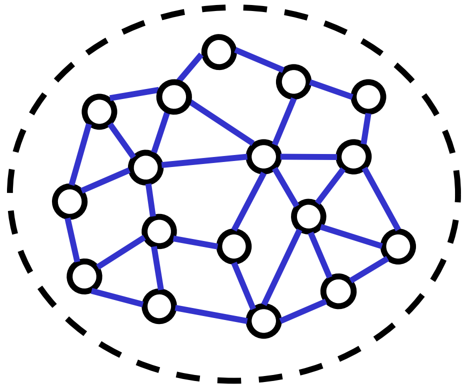


- the clustering coefficient is resistant to rewiring over a broad interval of p
  - it means that the small-world effect is hardly detectable locally: nodes continue seeing mostly the same “clique” of neighbors
- on the other hand, the average path length drops rapidly for low p
  - as soon as a few long-range “short-cut” connections are introduced, the original large-world starts collapsing
  - through a few bridges, far away cliques are put in contact and this is sufficient for a rapid spread of information

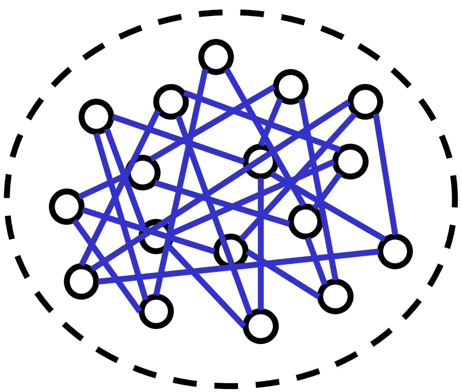
Four structural models
Small-world networks



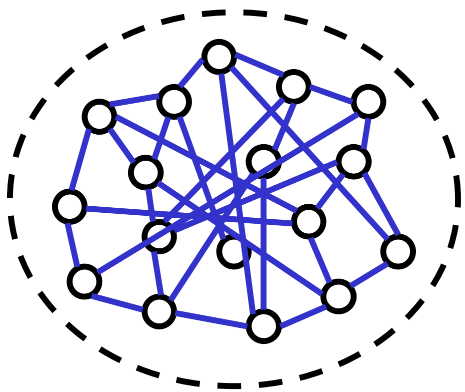
full, $\langle k \rangle = 16$



lattice, $\langle k \rangle = 3$

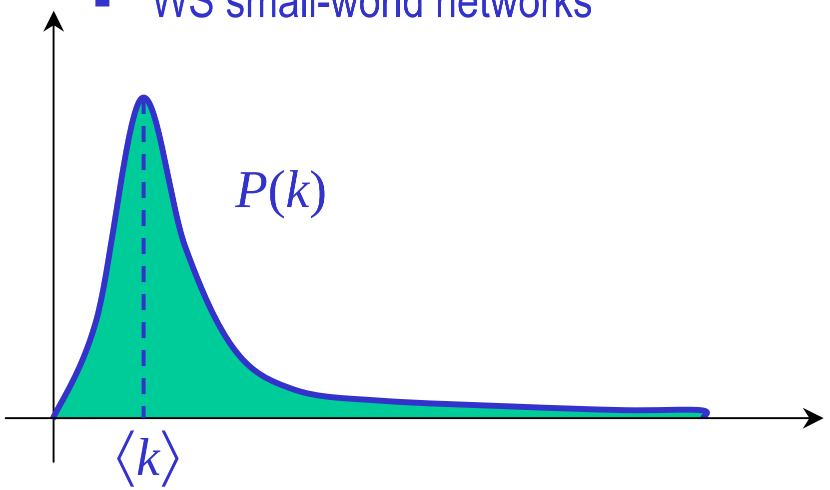


random, $\langle k \rangle = 3$

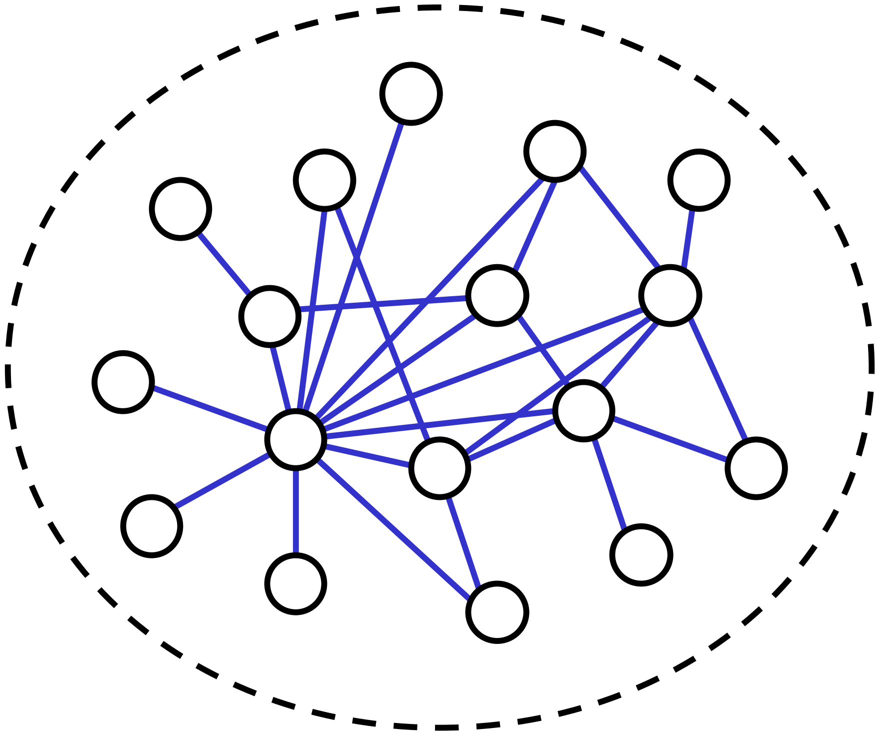


WS small-world, $\langle k \rangle = 3$

- on the other hand, the WS model still has a PEAK (Poisson) degree distribution (uniform connectivity)
- in that sense, it belongs to the same family of *exponential networks*:
  - fully connected networks
  - lattices
  - random graphs
  - WS small-world networks



Four structural models
Scale-free networks

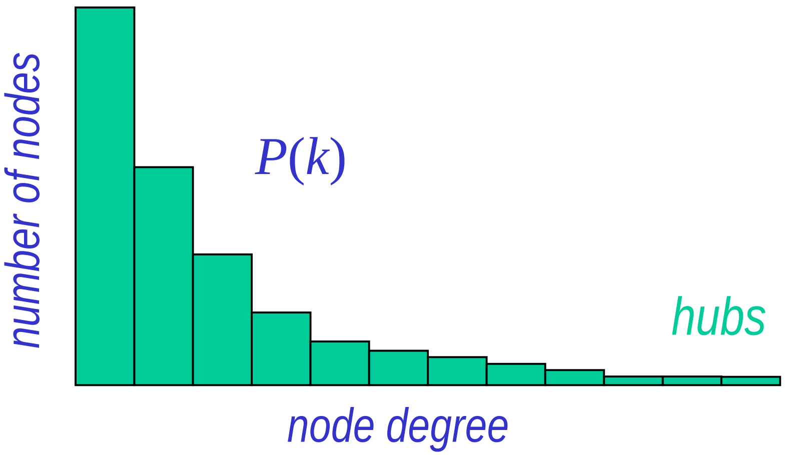


A schematic scale-free network

- in a *scale-free network* the degree distribution follows a POWER-LAW:

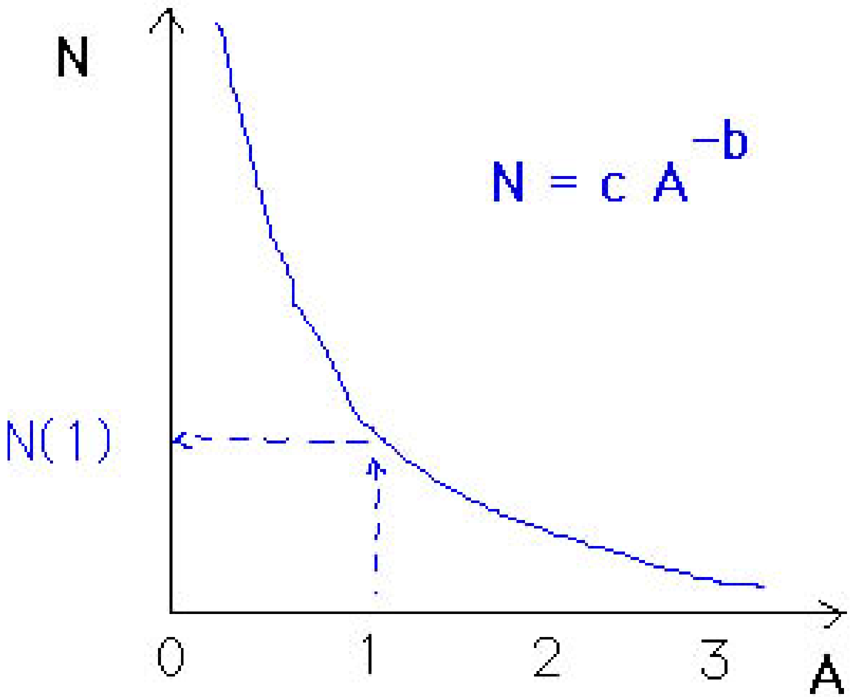
$$P(k) \sim k^{-\gamma}$$

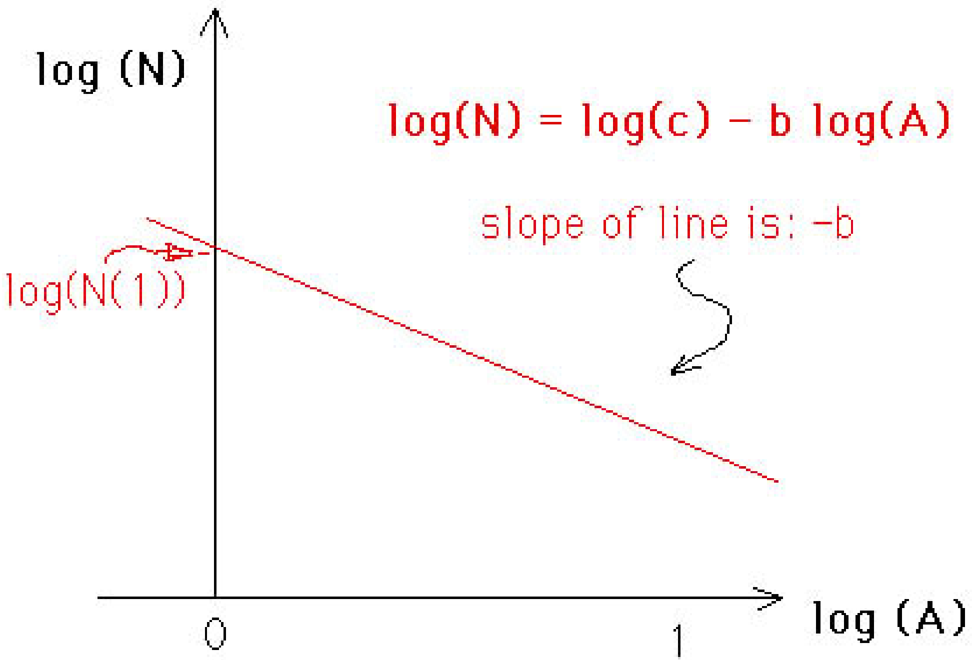
- there exists a small number of highly connected nodes, called *hubs* (tail of the distribution)
- the great majority of nodes have few connections (head of the distribution)



Four structural models
Scale-free networks

- hyperbola-like, in linear-linear plot
- straight line, in log-log plot



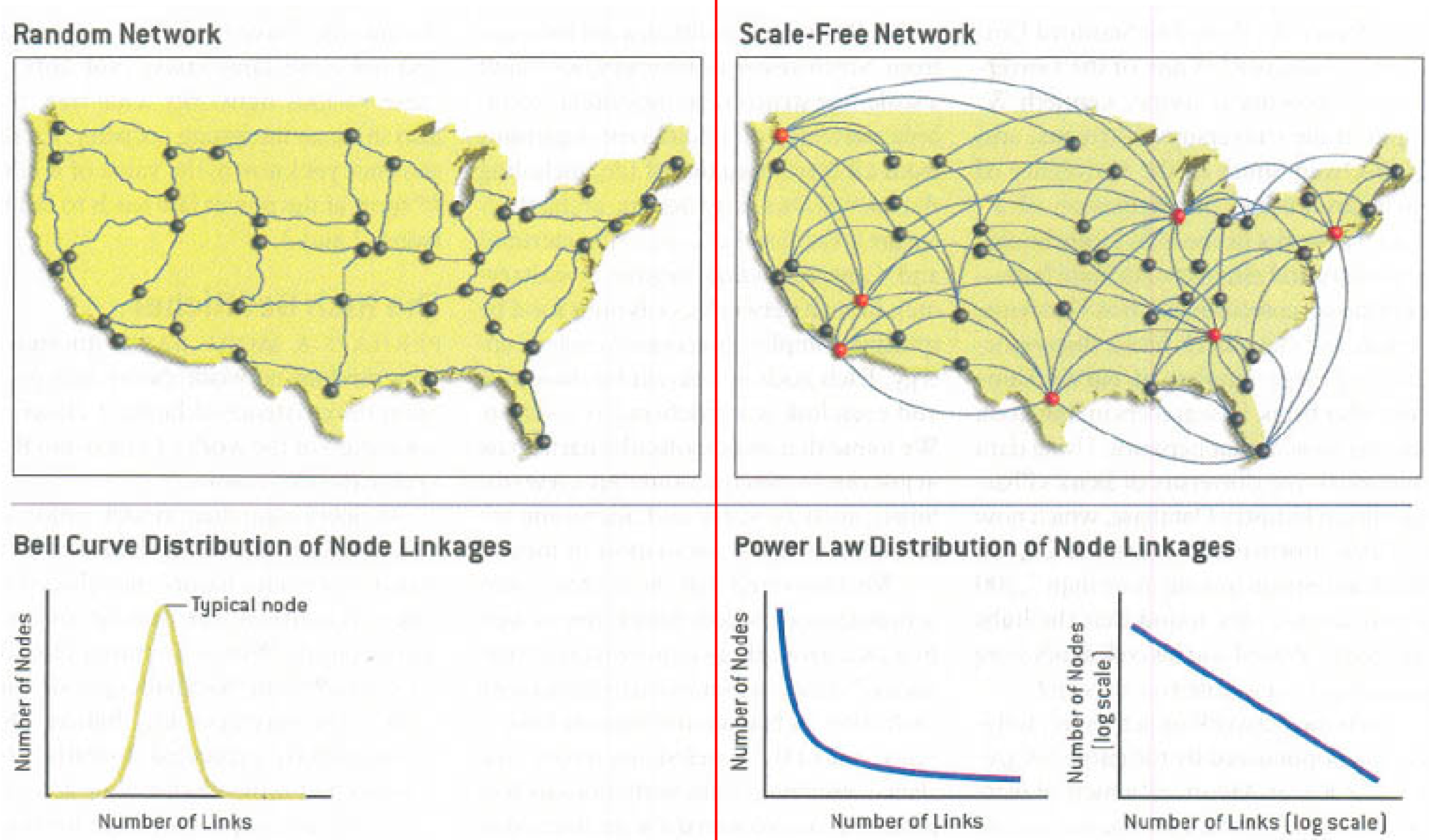


Typical aspect of a power law

(image from Larry Ruff, University of Michigan, http://www.geo.lsa.umich.edu/~ruff)

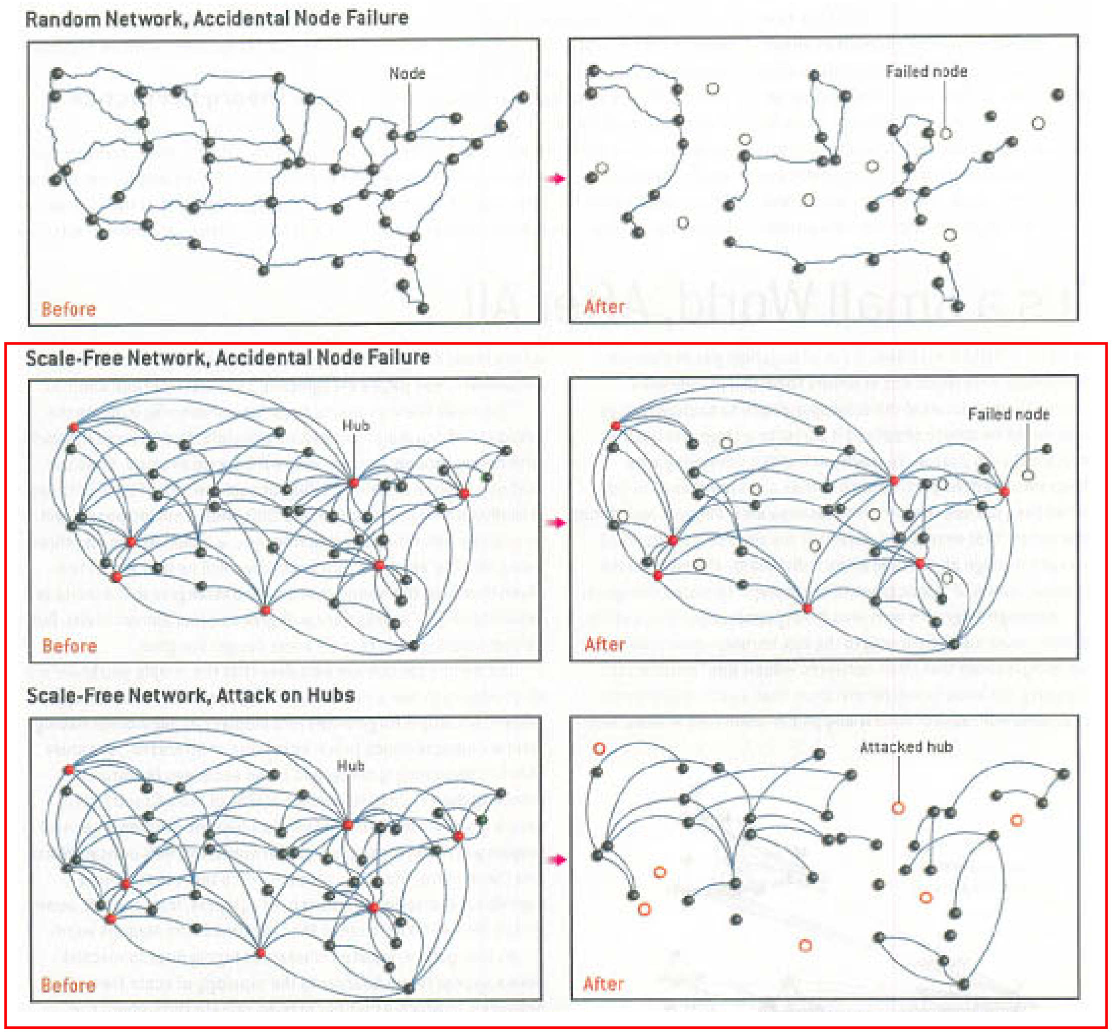
Four structural models
Scale-free networks

U.S. highway system *U.S. airline system*



(Barabási & Bonabeau, 2003)

Four structural models
Scale-free networks



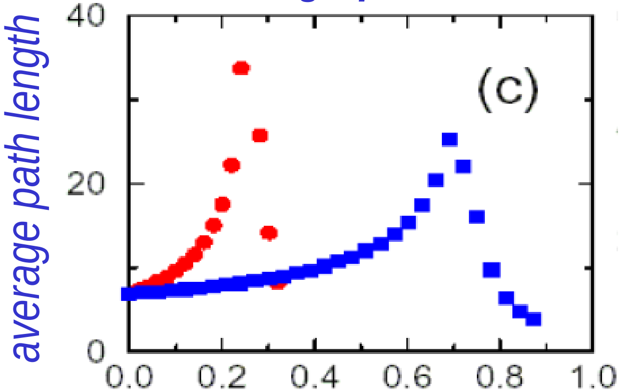
Effect of failures and attacks on scale-free networks

(Barabási & Bonabeau, 2003)

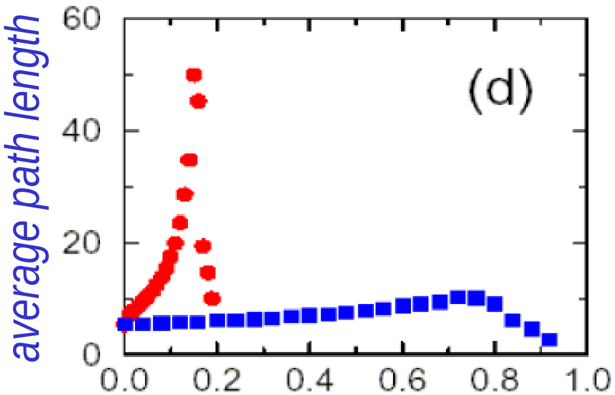
- regular networks are not resistant to random node failures: they quickly break down into isolated fragments
- scale-free networks are remarkably resistant to random accidental node failures . . .
- . . . however they are also highly vulnerable to targeted attacks on their hubs

Four structural models
Scale-free networks

Random graph



Scale-free network



random and targeted removal of nodes

(Albert, Jeong & Barabási, 1999)

- in a random graph the average path length increases significantly with node removal, then eventually breaks down

→ *for a while, the network becomes a large world*

- in a scale-free network, the average path length is preserved during ***random node removal***

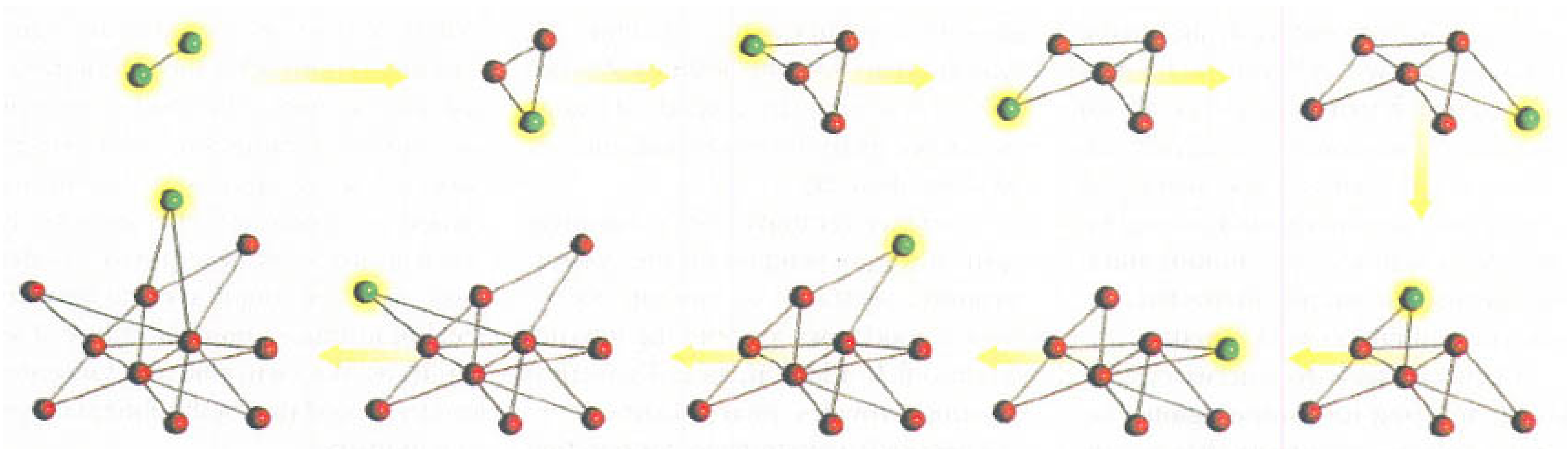
→ *it remains a small world*

- however, it fails even faster than a random graph under ***targeted removal***

Four structural models
Scale-free networks

- the *Barabási-Albert model*, reproduces the scale-free property by:
  - growth and
  - (linear) preferential attachment

- growth: a node is added at each step
- attachment: new nodes tend to prefer well-connected nodes ("the rich get richer" or "first come, best served") in linear proportion to their degree



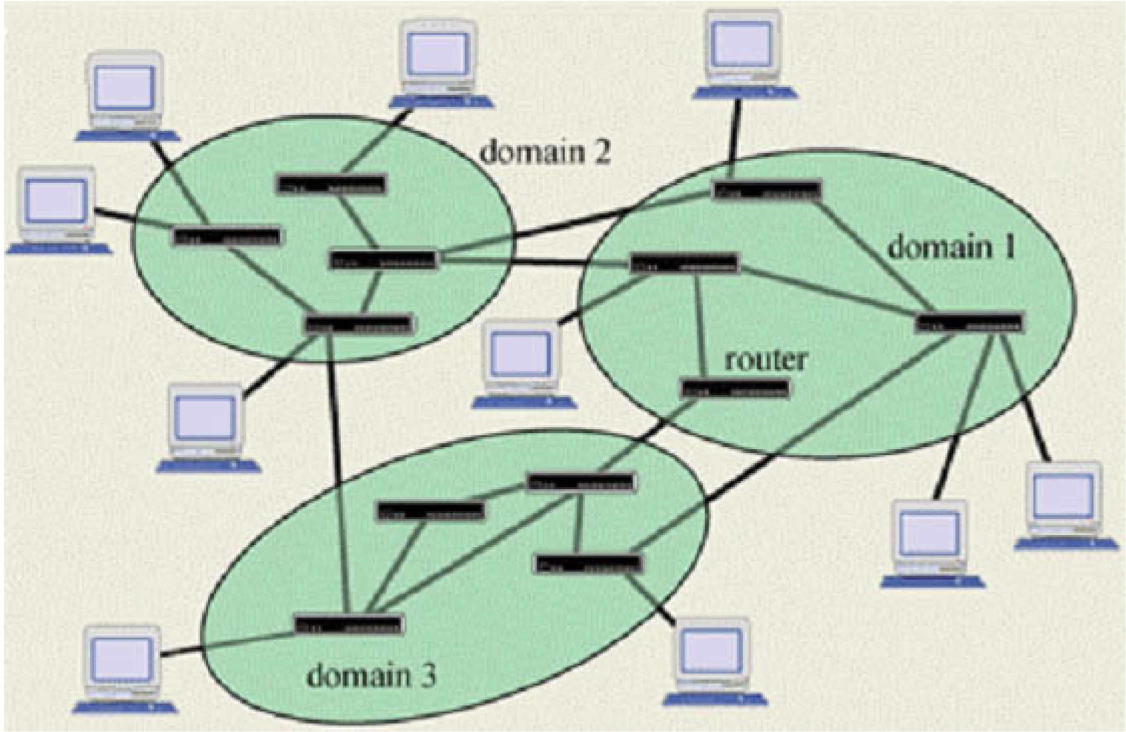
Growth and preferential attachment creating a scale-free network

(Barabási & Bonabeau, 2003)

Topology and Dynamics of Complex Networks

- Introduction
- Three structural metrics
- Four structural models
- Structural case studies
  - Internet
  - World Wide Web
  - Actors & scientists
  - Neural networks
  - Cellular metabolism
- Node dynamics and self-organization
- Bibliography

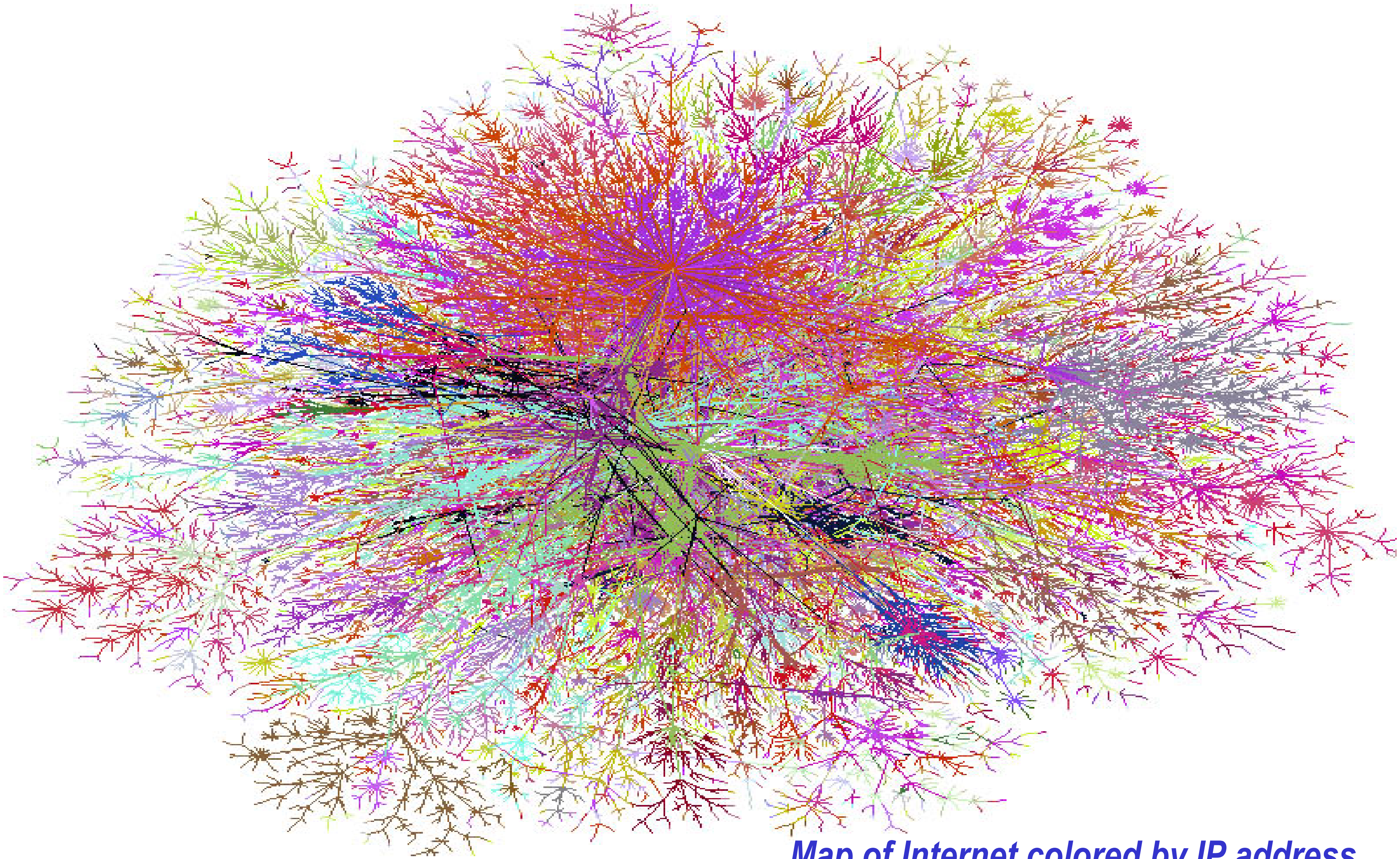
Structural case studies
Internet



Schema of the Internet
(Wang, X. F., 2002)

- the Internet is a network of routers that transmit data among computers
- routers are grouped into domains, which are interconnected
- to map the connections, “traceroute” utilities are used to send test data packets and trace their path

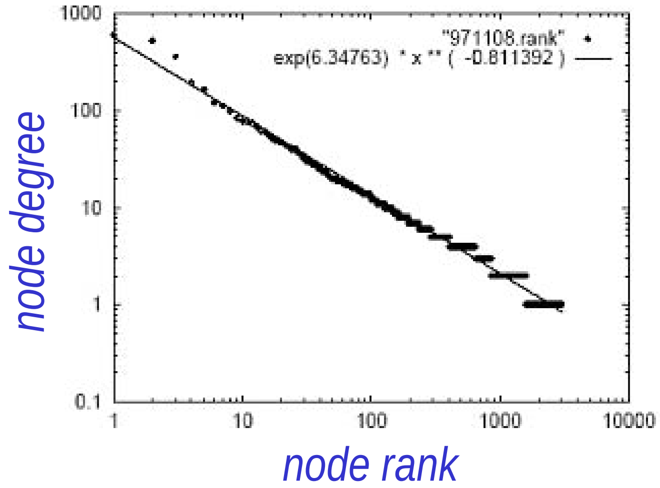
Structural case studies
Internet

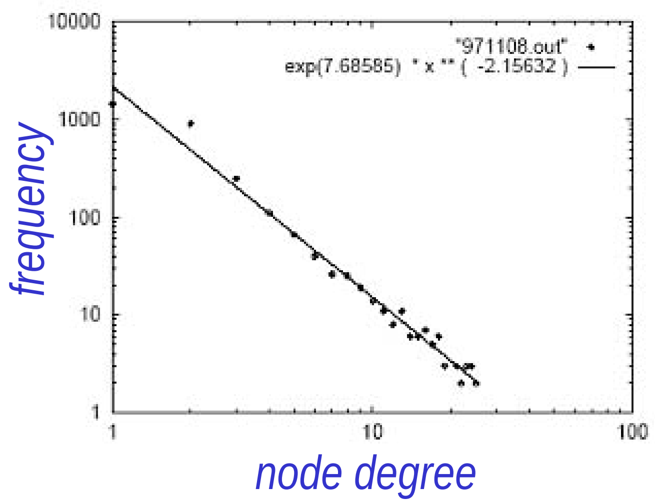


Map of Internet colored by IP address

(Bill Cheswick & Hal Burch, http://research.lumeta.com/ches/map)

Structural case studies
Internet





Two power laws of the Internet topology

(Faloutsos, Faloutsos & Faloutsos, 1999)

- the connectivity degree of a node follows a power of its rank (sorting out in decreasing order of degree):

  node degree ~ (node rank)$^{-\alpha}$

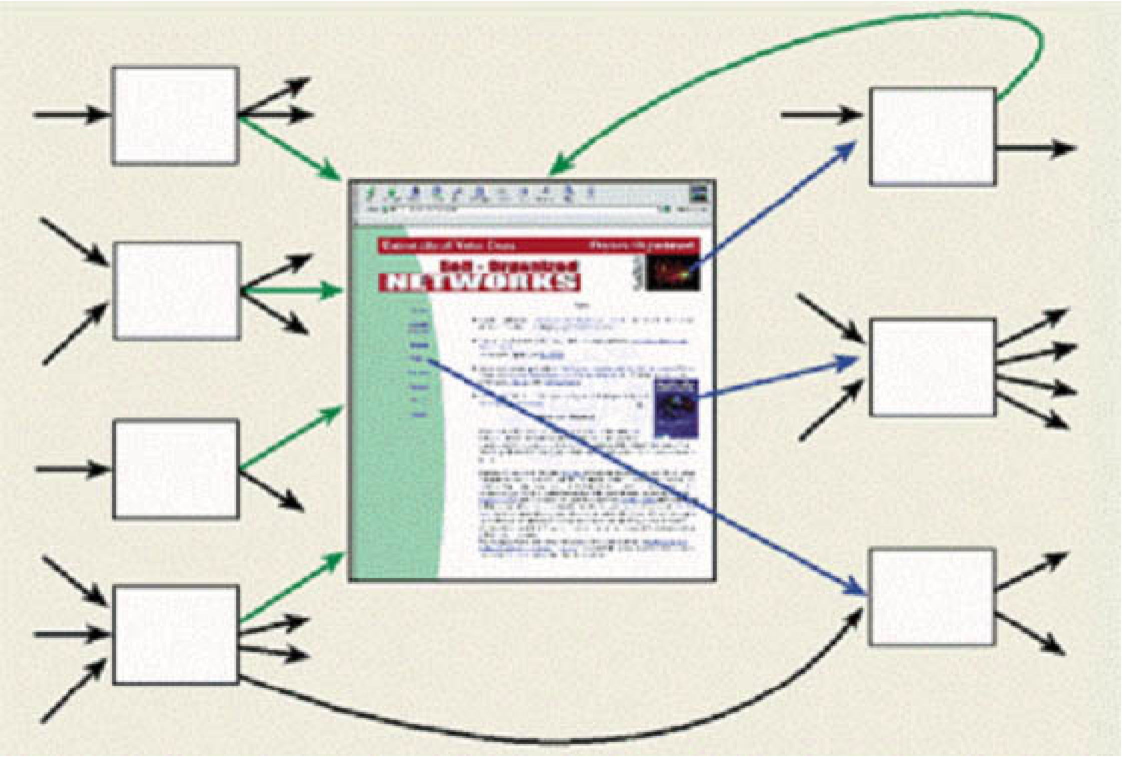
- the most connected nodes are the least frequent:

  degree frequency ~(node degree)$^{-\gamma}$

  $$P(k) \sim k^{-\gamma}$$

$\rightarrow$ *the Internet is a scale-free network*

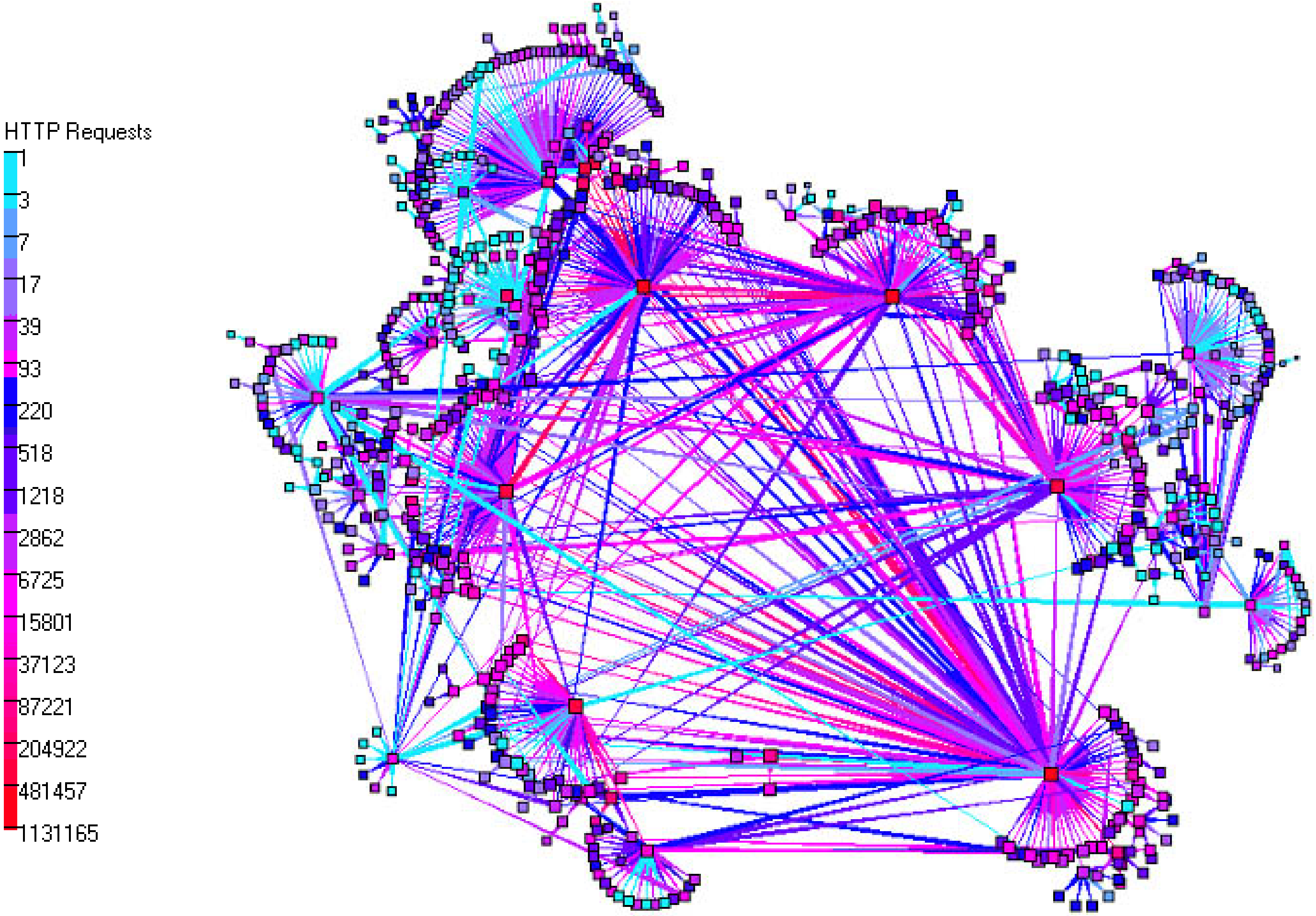
Structural case studies
World Wide Web



Schema of the World Wide Web of documents

- the World Wide Web is a network of documents that reference each other
- the nodes are the Web pages and the edges are the hyperlinks
- edges are directed: they can be outgoing and incoming hyperlinks

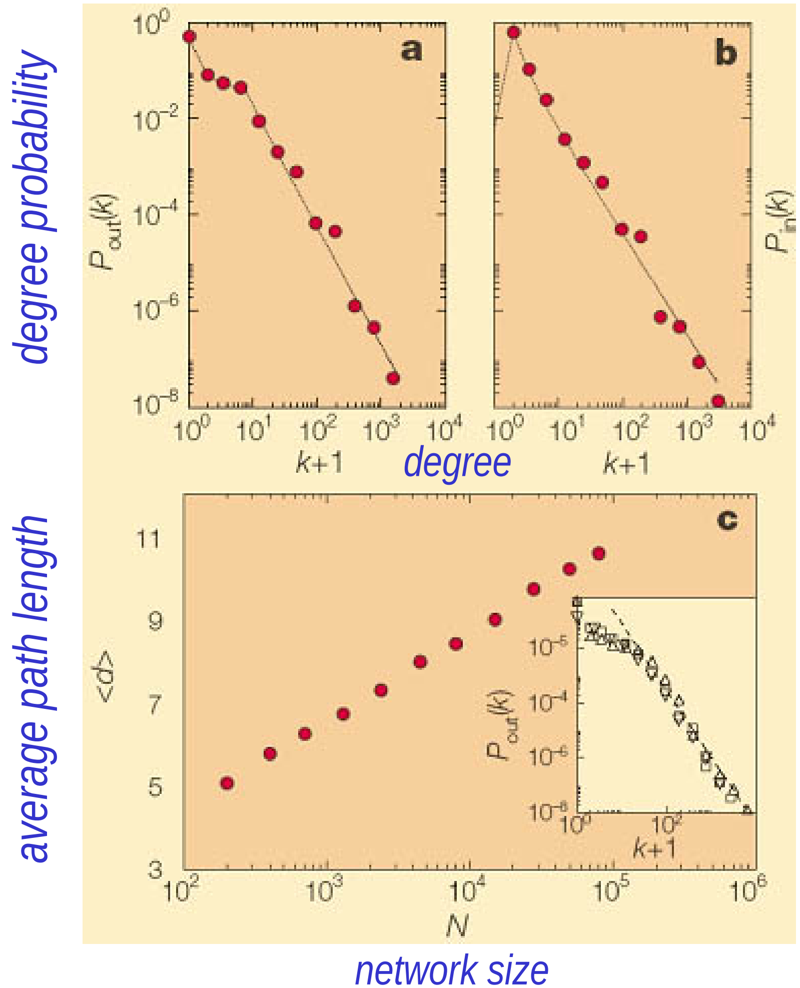
Structural case studies
World Wide Web



Hierachical topology of the international Web cache

(Bradley Huffaker, http://www.caida.org/tools/visualization/plankton)

Structural case studies
World Wide Web



Distribution of links on the World-Wide Web

(Albert, Jeong & Barabási, 1999)

- WWW is a scale-free network:

$$P(k) \sim k^{-\gamma}$$

with $\gamma_{out} = 2.45$ and $\gamma_{in} = 2.1$

- WWW is also a small world:

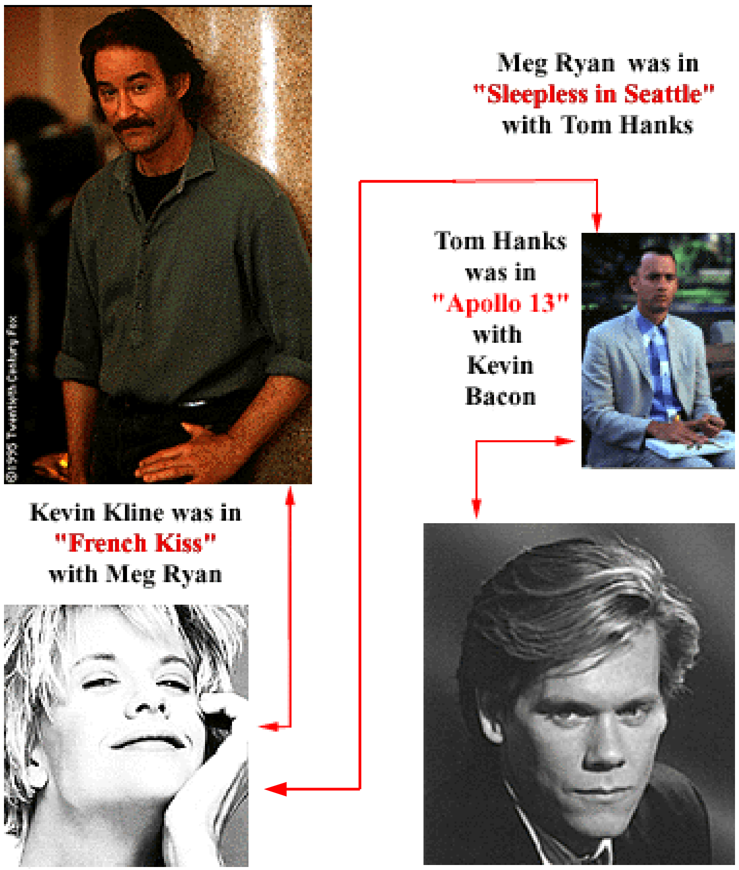
$$L \approx \alpha \ln N$$

with $L \approx 11$ for $N = 10^5$ documents

Structural case studies
Actors & scientists

"The Oracle of Bacon"

http://www.cs.virginia.edu/oracle



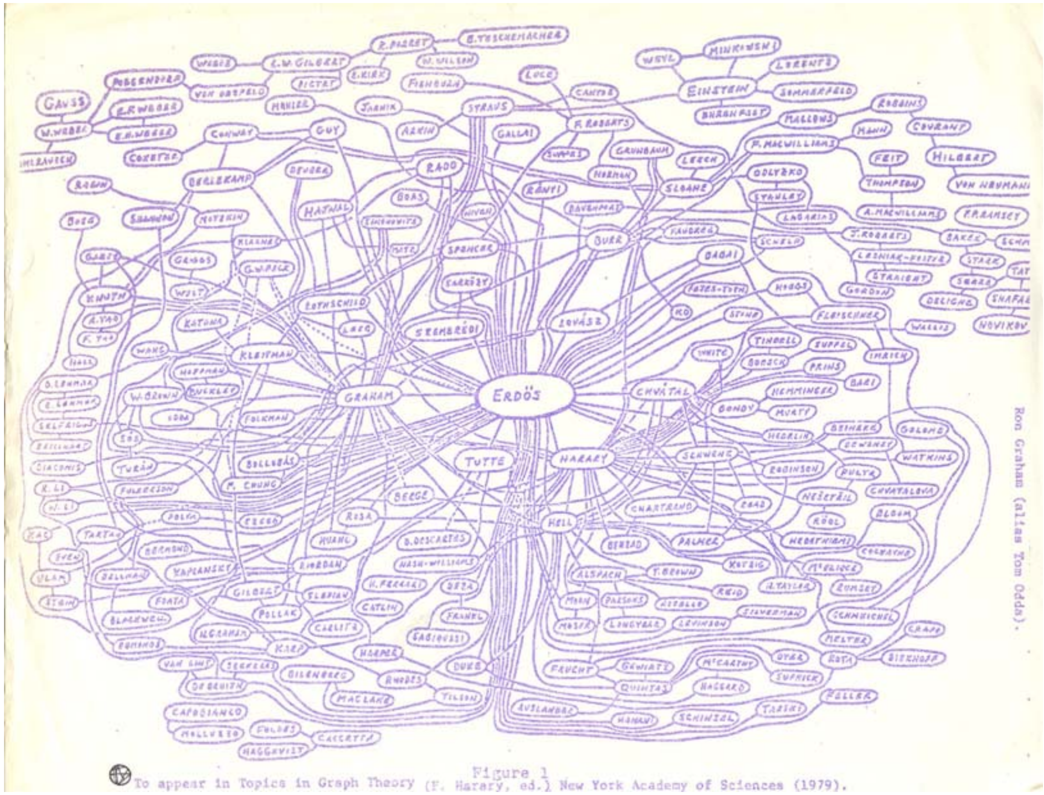
Path from K. Kline to K. Bacon = 3 (as of 1995)

(http://collegian.ksu.edu/issues/v100/FA/n069/fea-making-bacon-fuqua.html)

- a given actor is on average 3 movies away from Kevin Bacon (L_{Bacon}=2.946, as of June 2004) . . . or any other actor for that matter
- Hollywood is a small world
- . . . and it is a *scale-free* small world: a few actors played in a lot of movies, and a lot of actors in few movies

Structural case studies
Actors & scientists

“The Erdős Number Project”
http://www.oakland.edu/enp



Co-authors of Paul Erdős have number 1, co-authors of co-authors number 2, etc.

Mathematicians form a highly clustered ($C = 0.14$) small world ($L = 7.64$)

Topology and Dynamics of Complex Networks

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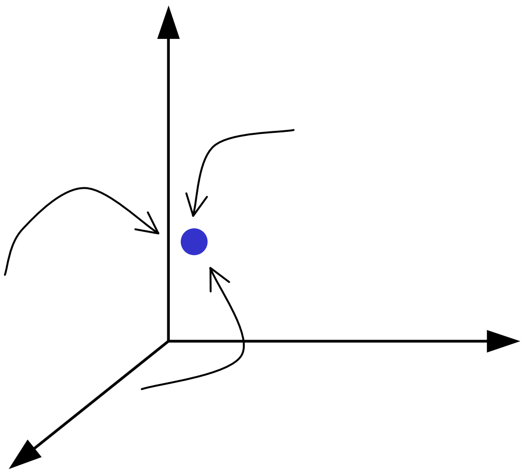
  - Node dynamics
  - Attractors in full & lattice networks
  - Synchronization in full networks
  - Waves in lattice networks
  - Epidemics in complex networks
- Bibliography

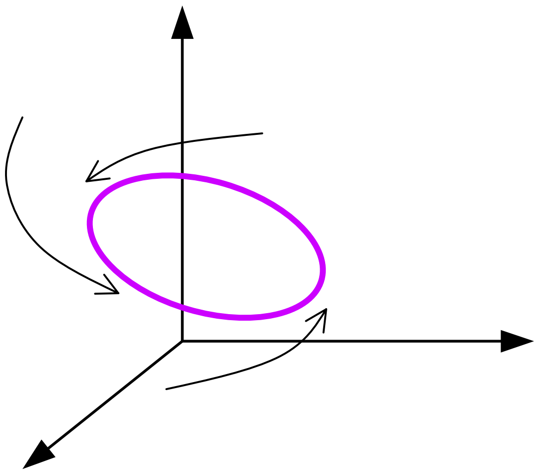
Node dynamics and self-organization

Node dynamics – *Individual node*

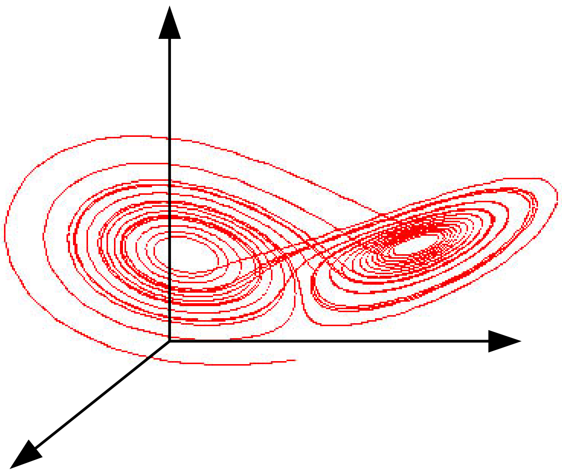
- each node in the network obey a differential equation: $\frac{dx}{dt} = f(x)$
- generally, three possible behaviors in phase space:



fixed point attractor



limit cycle attractor

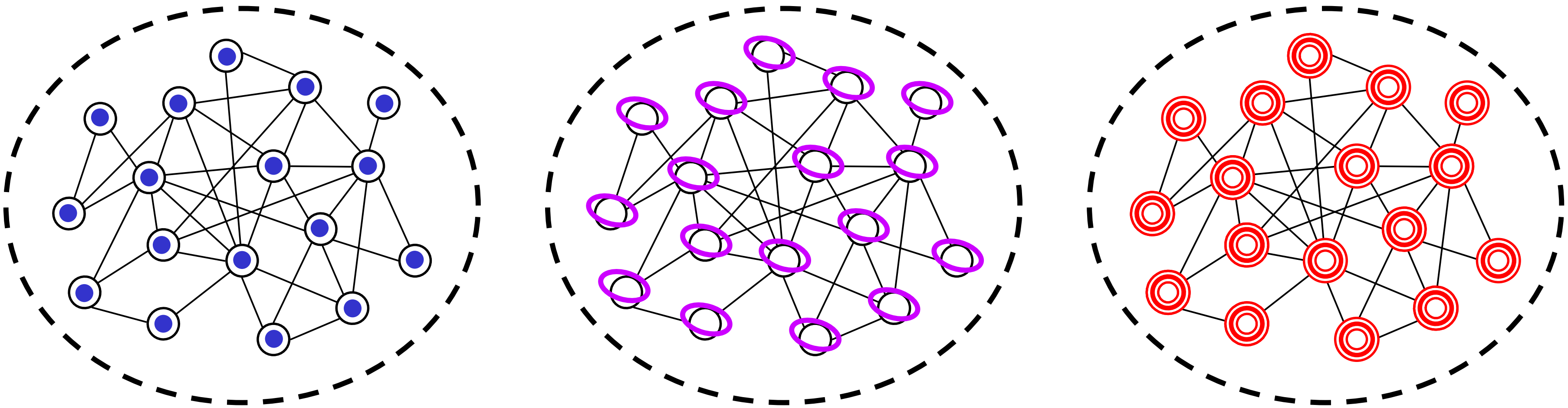


chaotic attractor

Node dynamics and self-organization

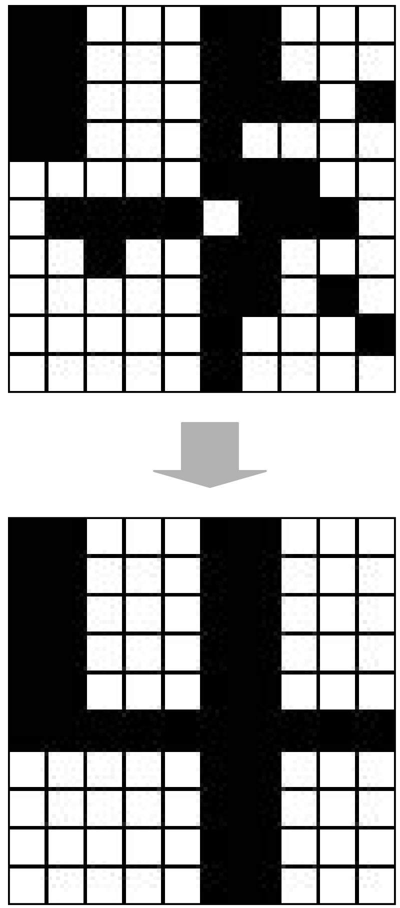
Node dynamics – *Coupled nodes*

- a complex network is a set of coupled nodes obeying: $\frac{dx_A}{dt} = f(x_A) + \sum_{A \leftarrow B} g(x_A, x_B)$
- generally, three types of complex network dynamics:

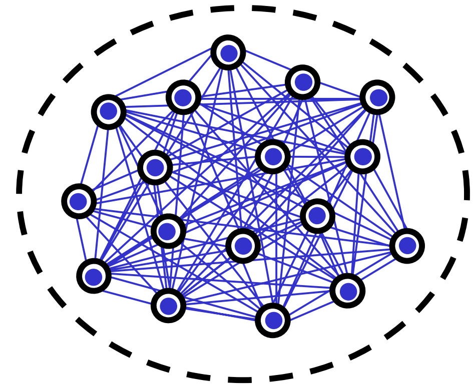


fixed point node network ***limit cycle node network*** ***chaotic node network***

Node dynamics and self-organization
Attractors in full networks



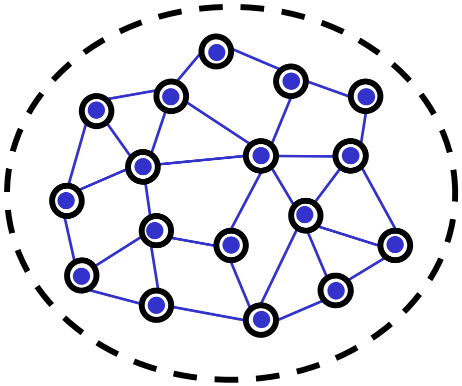
Pattern retrieval in Hopfield memory:
full graph with Ising-type interactions



- fixed point nodes
- fully connected network

→ *a few fixed patterns* (≈ 0.14 N)

Node dynamics and self-organization
Attractors in lattice networks



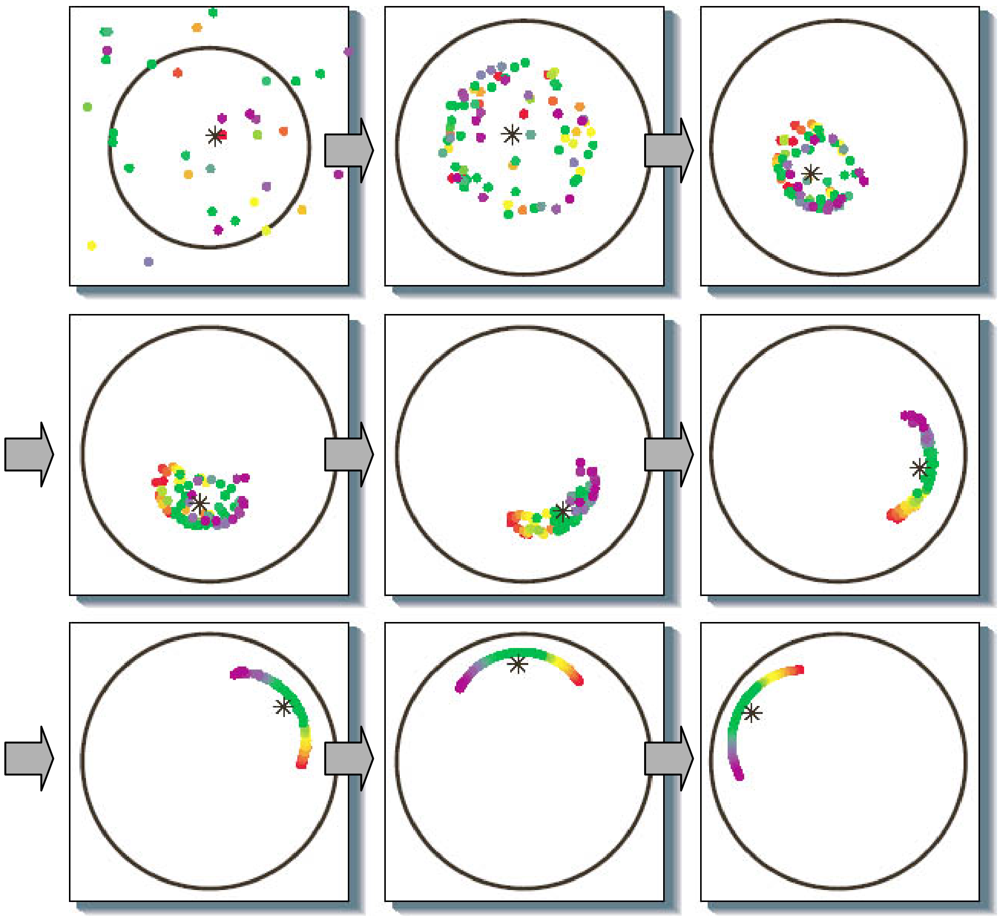
- fixed point nodes
- regular lattice network

$\rightarrow$ *a great number of new patterns*

Pattern formation in animal pigmentation: 2-D lattice with stationary reaction-diffusion

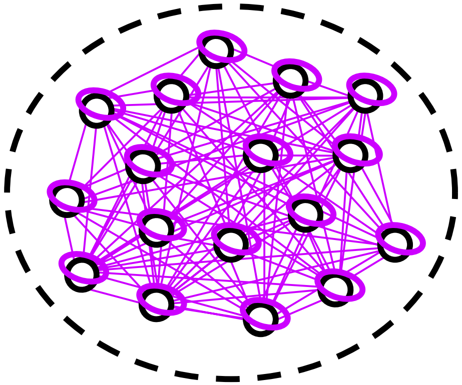
(NetLogo simulation, Uri Wilensky, Northwestern University, IL)

Node dynamics and self-organization
Synchronization in full networks



Spontaneous synchronization in a network of limit-cycle oscillators with distributed natural frequencies

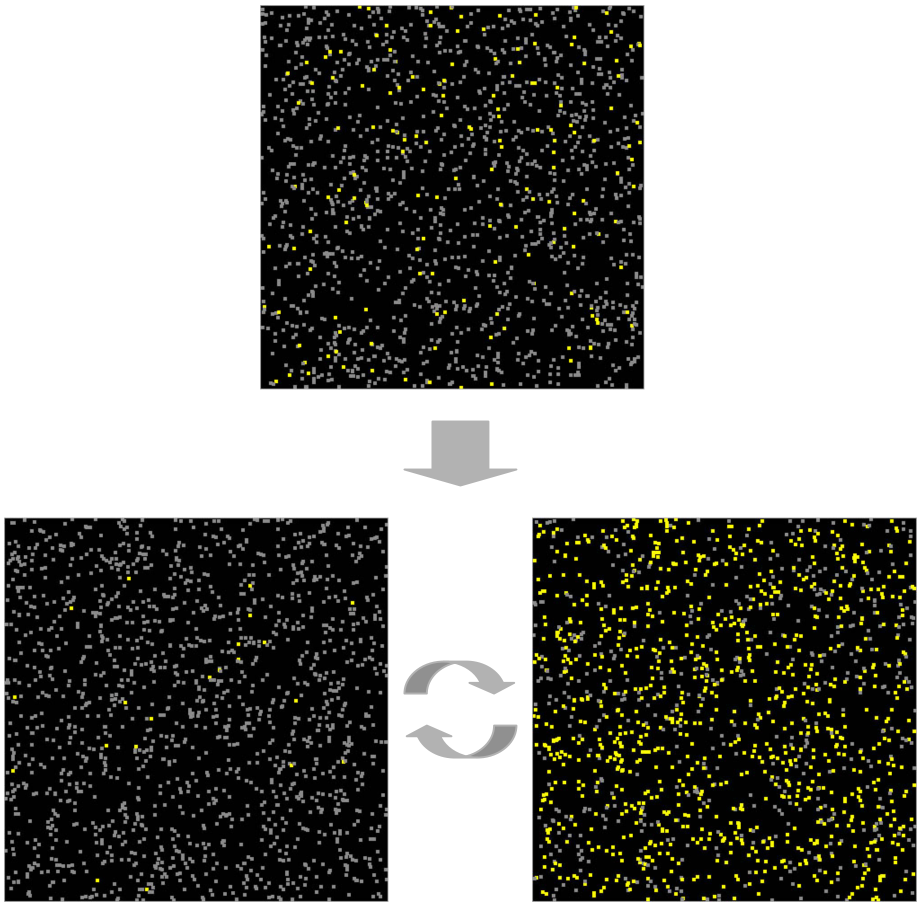
(Strogatz, 2001)

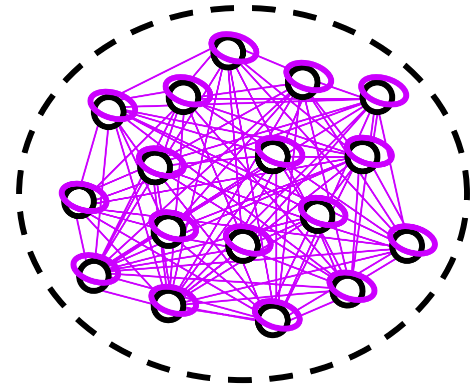


- limit cycle nodes
- fully connected network

→ *global synchronization*

Node dynamics and self-organization
Synchronization in full networks





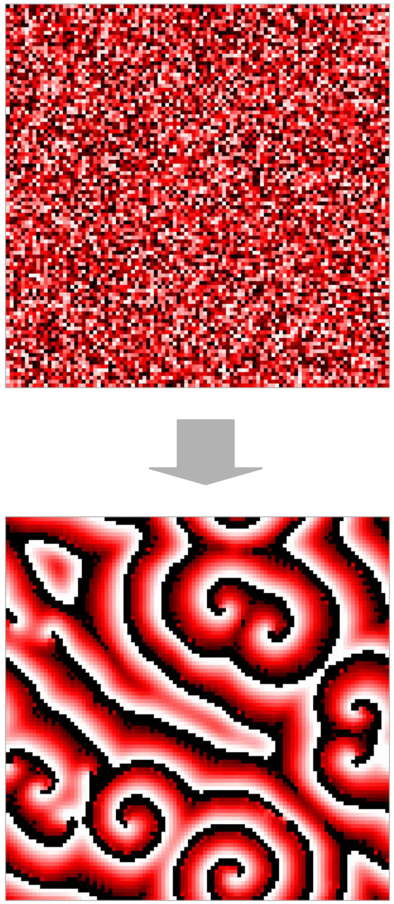
- limit cycle nodes
- fully connected network

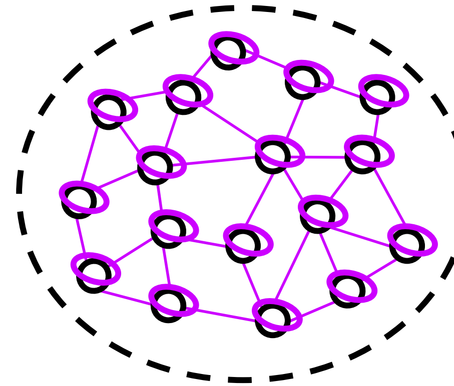
→ *global synchronization*

Spontaneous synchronization in a swarm of fireflies:
(almost) fully connected graph of independent oscillators

(NetLogo simulation, Uri Wilensky, Northwestern University, IL)

Node dynamics and self-organization
Waves in lattice networks





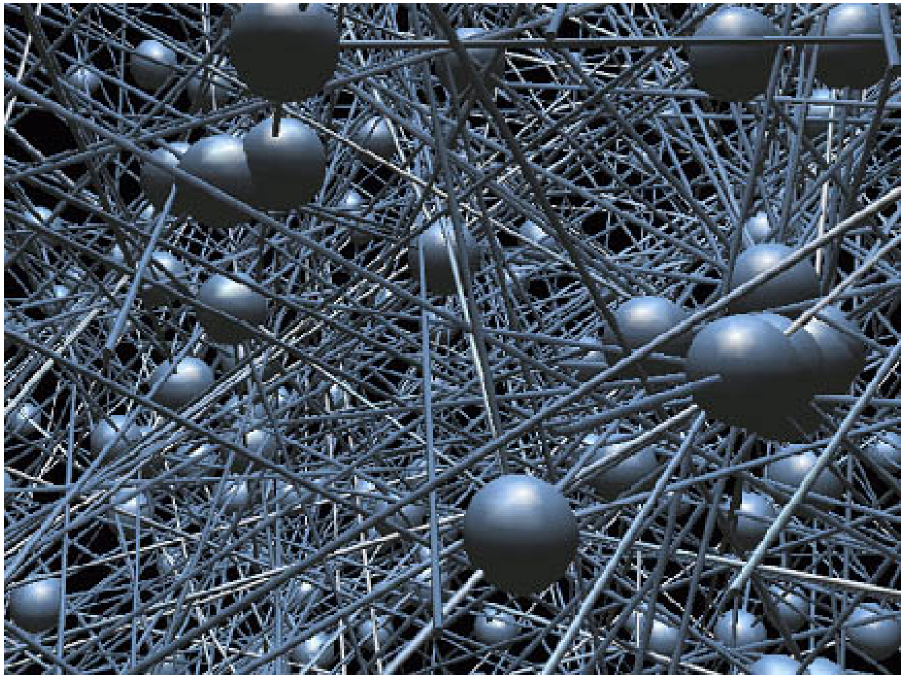
- limit cycle nodes
- regular lattice network

→ *traveling waves*

BZ reaction or slime mold aggregation:
2-D lattice with oscillatory reaction-diffusion

(NetLogo simulation, Uri Wilensky, Northwestern University, IL)

Node dynamics and self-organization
Epidemics in complex networks

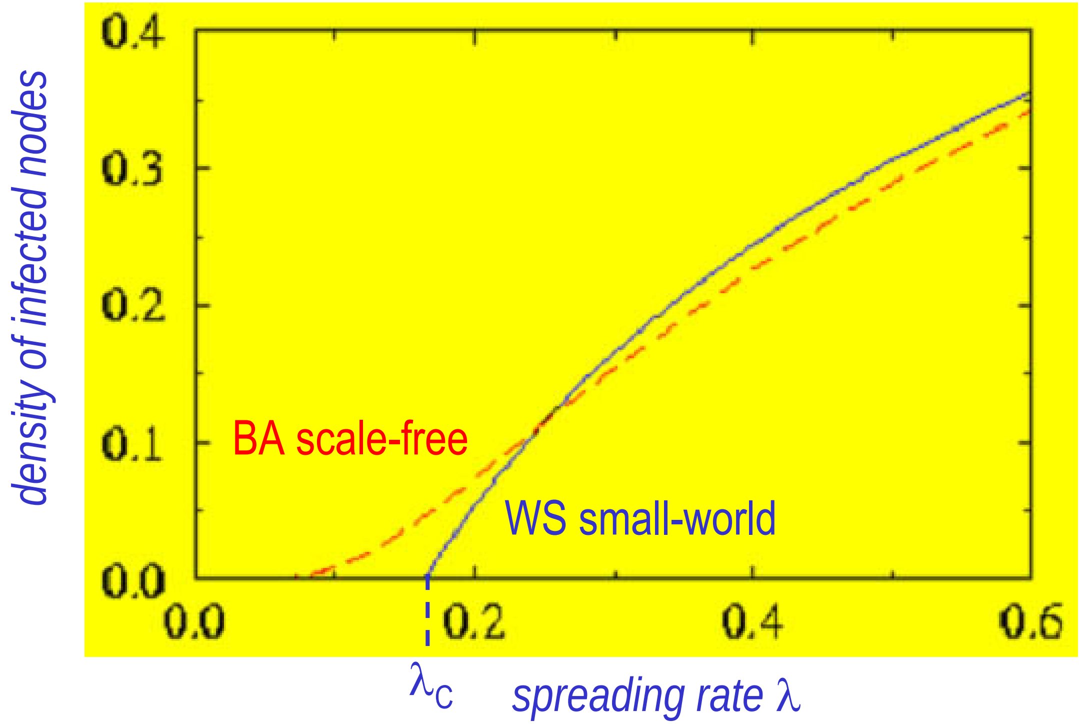


3-D visualization of social links
(A. S. Klovdahl, http://carnap.ss.uci.edu/vis.html)

- understand of beneficial or nefarious activity/failures spread over a network:
  - diseases
  - power blackouts
  - computer viruses
  - fashions, etc.
- *susceptible-infected-susceptible (SIS)* epidemiological model:
  - two node states: infected or susceptible
  - susceptible nodes can get infected with probability ν
  - infected nodes heal and become susceptible again with proba δ

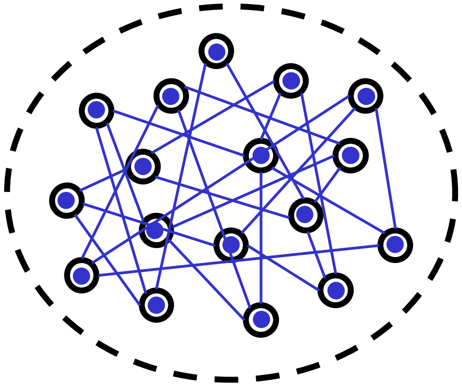
$\rightarrow$ *spreading rate:* $\lambda = \nu / \delta$

Node dynamics and self-organization
Epidemics in complex networks



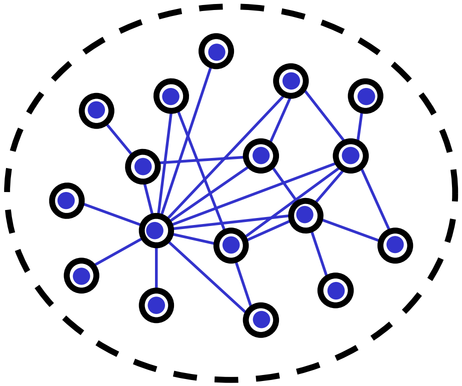
Epidemic on exponential and scale-free networks

(Pastor-Satorras & Vespignani, 2001)



- exponential network

$\rightarrow$ *spread with threshold*



- scale-free network

$\rightarrow$ *spread WITHOUT threshold*

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  - Studies

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